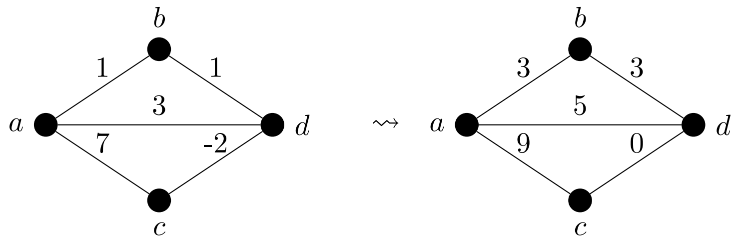
Institut für Theoretische Informatik
Leibniz Universität Hannover

Solution to exercise sheet 3

16.04.2013

Exercise 1: Consider the following modification of the DJP algorithm to work with negative weights: Determine the smallest weight $c \in \mathbb{Z}$ in the weighted graph $G = (V, E, w)$, i.e., the edge e s.t. $w(e) = c$. Then for all edges $f \in E$ set $w'(f) := w(f) - c$. Then $G' = (V, E, w')$ has no negative weights.
Does the DJP algorithm work correctly on this type of graph? Prove your claim.

Solution: Now we claim that DJP does not work correctly on G' because this modification does not maintain the shortest path property, i.e., if π was a shortest path in G from s to t, then π is a shortest s-t-path in G'. The following counter-example proves this.



In the left graph a, b, d is the shortest path from a to d. In the right graph a, d is shorter. □

Exercise 2: Prove the upper-bound property:
Let $G = (V, E, w)$ be a weighted digraph and $s \in V$ be a vertex. Then $\text{cost}[v] \geq d(s, v)$ for all $v \in V$ and this invariant is maintained over any sequence of relaxation steps on the edges of G. Moreover, once $\text{cost}[v]$ achieves its lower bound $d(s, v)$, it never changes.

So prove the invariant '$\text{cost}[v] \geq d(s, v)$ for all $v \in V$' by induction over the number of relaxation steps.

Solution: For the basis, $\text{cost}[v] \geq d(s, v)$ is certainly true after initialization, since $\text{cost}[s] = 0 \geq d(s, s)$ (note that $d(s, s) = -\infty$ if s in on a negative-weight cycle and 0 otherwise) and $\text{cost}[v] = \infty$ implies $\text{cost}[v] \geq d(s, v)$ for all $v \in V - \{s\}$.
For inductive step, consider the relaxation of an edge (u, v). By the inductive hypothesis, $\text{cost}[x] \geq d(s, x)$ for all $x \in V$ prior to the relaxation. The only $\text{cost}[\cdot]$ value that may change is $\text{cost}[v]$. If it changes, we have

$$\begin{aligned}
\text{cost}[v] &= \text{cost}[u] + w(u, v) & \\
&\geq d(s, u) + w(u, v) & \text{(IH)} \\
&\geq d(s, v) & \text{(triangle property)}
\end{aligned}$$

and so the invariant is maintained.
To see that the value of $\text{cost}[v]$ never changes once $\text{cost}[v] = d(s, v)$, note that having achieved its lower bound, $\text{cost}[v]$ cannot decrease because we have just shown that $\text{cost}[v] \geq d(s, v)$, and it cannot increase because relaxation steps do not increase $\text{cost}[\cdot]$ values. □

Exercise 3: Prove the convergence property:
Let $G = (V, E, w)$ be a weighted digraph and let $(u, v) \in E$. Then, immediately after relaxing edge (u, v) in the **if** block we have $\text{cost}[v] \leq \text{cost}[u] + w(u, v)$.

Solution: If, just prior to relaxing edge (u, v), we have $\text{cost}[v] > \text{cost}[u] + w(u, v)$, then $\text{cost}[v] = \text{cost}[u] + w(u, v)$ afterward. If, instead, $\text{cost}[v] \leq \text{cost}[u] + w(u, v)$ just before the relaxation, then neither $\text{cost}[u]$ nor $\text{cost}[v]$ changes, and so $\text{cost}[v] \leq \text{cost}[u] + w(u, v)$ afterward. □

Exercise 4: Prove the path-relaxation property:
If $\pi = (v_0, v_1, \dots, v_k)$ is a shortest path form $s = v_0$ to v_k, and the edges of p are relaxed in the order $(v_0, v_1), (v_1, v_2), \dots, (v_{k-1}, v_k)$, then $\text{cost}[v_k] = d(s, v_k)$. This property holds regardless of any other relaxation steps that occur, even if they are intermixed with relaxations of the edges of π.

Show by induction that after the ith edge of π is relaxed, we have $\text{cost}[v_i] = d(s, v_i)$.

Solution: For the basis, $i = 0$, and before any edge of π have been relaxed, we have form the initialization that $\text{cost}[v_0] = \text{cost}[s] = 0 = d(s, s)$. By the upper-bound property, the value of $\text{cost}[s]$ never changes after initialization.
For the inductive step, we assume that $\text{cost}[v_{i-1}] = d(s, v_{i-1})$, and we examine the relaxation of edge (v_{i-1}, v_i). By the convergence property, after this relaxation, we have $\text{cost}[v_i] = d(s, v_i)$, and this equality is maintained at all times thereafter. □

Exercise 5: Prove the no-path property:
Suppose that in a weighted, digraph $G = (V, E, w)$, no path connects a source $s \in V$ to a given vertex $v \in V$. Then we have $\text{cost}[v] = d(s, v) = \infty$ after initialization, and this equality is maintained as an invariant over any sequence of relaxation steps on the edges of G.

Solution: By the upper-bound property, we always have $\infty = d(s, v) \leq \text{cost}[v]$, and thus $\text{cost}[v] = \infty = d(s, v)$. □

Exercise 6: Prove the cost-array correctness:
Let $G = (V, E, w)$ be a weighted digraph and $s \in V$, and assume that G contains no negative cycles reachable from s. Then, after $|V| - 1$ iterations of the **for** loop it holds $\text{cost}[v] = d(s, v)$ for all vertices v that are reachable from s.

Solution: We prove the claim by appealing to the path-relaxation property. Consider any vertex v that is reachable from s, and let $\pi = (v_0, v_1, \dots, v_k)$ where $v_0 = s$ and $v_k = v$, be any acyclic shortest path from s to v. Path π has at most $|V| - 1$ edges, and so $k \leq |V| - 1$. Each of the $|V| - 1$ iterations of the **for** loop relaxes all E edges. Among the edges relaxed in the ith iteration, for $i = 1, 2, \ldots, k$, is (v_{i-1}, v_i). By the path-relaxation property, therefore, $\mathrm{cost}[v] = \mathrm{cost}[v_k] = d(s, v_k) = d(s, v)$. $\square$

Exercise 7: Prove the predecessor-subgraph property:
Let $G = (V, E, w)$ be a weighted digraph, $s \in V$, and assume G contains no negative-weight cycles reachable from s. After the initialization of the algorithm execute any sequence of relaxation steps producing $\mathrm{cost}[v] = d(s, v)$ for all $v \in V$. Then, the predecessor subgraph $G_{\mathrm{route}[\cdot]}$ is as shortest-paths tree rooted at s.

Solution: We must check the three properties of Observation 6 for shortest-paths trees. For the first property, show that $V_{\mathrm{route}[\cdot]}$ is the set of vertices reachable from s. By definition, a shortest-path weight $d(s, v)$ is finite iff v is reachable from s, hence the vertices that are reachable from s are exactly those with finite $\mathrm{cost}[\cdot]$ values. But a vertex $v \in V - \{s\}$ has been assigned a finite value for $\mathrm{cost}[v]$ iff $\mathrm{route}[v] \neq -$. Hence, the vertices in $V_{\mathrm{route}[\cdot]}$ are exactly the reachable ones. Property two follows directly from Lemma 1.8.
It remains to prove the last property of shortest-paths trees: for each vertex $v \in V_{\mathrm{route}[\cdot]}$ the unique simple path $\pi = v_0, \ldots, v_k$ with $s = v_0$ and $v = v_k$ in $G_{\mathrm{route}[\cdot]}$ is a shortest path from s to v in G. For $1 \leq i \leq k$ we have both $\mathrm{cost}[v_i] = d(s, v_i)$ and $\mathrm{cost}[v_i] \geq \mathrm{cost}[v_{i-1}] + w(v_{i-1}, v_i)$ from which we conclude $w(v_{i-1}, v_i) \leq d(s, v_i) - d(s, v_{i-1})$. Summing the weights along π yields

$$
\begin{aligned}
w(\pi) &= \sum_{i=1}^{k} w(v_{i-1}, v_i) \\
&\leq \sum_{i=1}^{k} \big(d(s, v_i) - d(s, v_{i-1})\big) \\
&= d(s, v_k) - d(s, v_0) && \text{(sum telescopes)} \\
&= d(s, v_k) && (d(s, v_0) = d(s, s) = 0).
\end{aligned}
$$

Thus, $w(\pi) \leq d(s, v_k)$. Since $d(s, v_k)$ is a lower bound on the weight of any path from s to v_k, we conclude that $w(\pi) = d(s, v_k)$, hence π is a shortest path from s to $v_k = v$. $\square$