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Solution to exercise sheet 1

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Exercise 1: Prove Observation 1: Every forest is bipartite.

Solution: Let $F = (V, E)$ be a forest. Then F is acyclic. Divide V into its connected components (i.e., trees) $V = V_1 \uplus \cdots \uplus V_k$. In such a component there exists a unique path from one vertex to another. Now define for each component V_i and an arbitrary vertex $u_i \in V_i$ a function $d_i \colon V_i \times V_i \to \mathbb{N}$ such that $d_i(u_i, v) := |\pi|$ for the u_i-v-path π. Now we are able to define the bipartition of V into A and B as follows:

$$A := \bigcup_{i=1}^{k} \{v \in V_i \mid d_i(u_i, v) \equiv_2 0\} \cup \{u_i\},$$
$$B := \bigcup_{i=1}^{k} \{v \in V_i \mid d_i(u_i, v) \equiv_2 1\}.$$

Let $u, v \in A$ be some nodes. Assume for contradiction that $(u, v) \in E$. As $u, v \in V_i$ for some $1 \leq i \leq k$ we have that $d_i(u_i, v) \neq d_i(u_i, u)$ because $u \neq v$ and V_i is a tree. W.l.o.g. $d_i(u_i, v) \equiv_2 0$ and then $d_i(u_i, u) \equiv 21$. Hence $u \in B$ and $v \in A$ which is a contradiction. $\square$

Exercise 2: Prove Observation 2:
Let e be an edge of graph G. Then the following claims are equivalent.

1. e is a bridge.
2. e is not part of any circle in G.
3. It holds $\kappa(G) + 1 = \kappa(G - e)$.

Solution: We will prove the circular argument. Let $G = (V, E)$ be a graph.
$1 \Rightarrow 2$: Let e be a bridge. Assume that $e = \{u, v\}$ is part of a circle in G. Consider $V' \subseteq V$ as the connected component inside which π is located. Hence for all vertices $w \in V'$ it holds that there are v-w-paths. Observe that in the Graph $G - e$ there is still a u-v-path through the remainder of the previous circle. Hence $\kappa(G) = \kappa(G - e)$ which is a contradiction.
$2 \Rightarrow 3$: Now e is not part of any circle in G. Let $e = \{u, v\}$. Hence there exists no u-v-path π such that $\{u, v\} \notin \pi$ (because otherwise we would have a circle with e). Hence $G - e$ contains no u-v-path wherefore $\kappa(G - e) = \kappa(G) + 1$.
$3 \Rightarrow 1$: follows by definition of a bridge. $\square$

Exercise 3: Let $G = (V, E)$ be an undirected connected graph and π_1, π_2 two longest simple paths in G. Prove that $\pi_1 \cap \pi_2 \neq \emptyset$, where

$$\pi_1 \cap \pi_2 := \{ v \in V \mid v \in \pi_1 \text{ and } v \in \pi_2 \} .$$

Solution: Given the undirected connected graph $G = (V, E)$ and two longest simple paths $\pi_1 = p_1, \ldots, p_n$ and $\pi_2 = q_1, \ldots, q_n$ we observe that $|\pi_1| = |\pi_2|$ holds. Assume for contradiction that there is no common vertex in π_1 and π_2.
As G is connected we can construct a path π from the initial node p_1 of π_1 to the initial node q_1 of π_2. π must leave π_1 on some vertex x. Hence x divides π into two pieces. Lets say the longer part is ρ_1 and is the path from either the first or the last vertex to x. π also divides π_2 and defines a longer subpath in π_2 which will be denoted with ρ_3 and moves from the first or the last node of π_2 to the vertex y where π meets π_2.
The subpath in π which is between x and y is denoted with ρ_2. As π_1 and π_2 have no common vertex (by assumption) ρ_2 must consist of at least one edge. Through ρ_1, ρ_2, and ρ_3 we define a new path π'. Through the previous choice of the longer subpaths it must hold

$$|\rho_1| \geq \frac{|\pi_1|}{2} \text{ and } |\rho_3| \geq \frac{|\pi_2|}{2}.$$

Combining with $|\rho_2| \geq 1$ we get for π':

$$|\pi'| \geq \frac{|\pi_1|}{2} + 1 + \frac{|\pi_2|}{2} = |\pi_1| + 1 = |\pi_2| + 1.$$

As π_1, π_2 are two longest paths in G we have a contradiction because π' is longer. Hence there must be a common vertex. $\square$