Institut für Theoretische Informatik
Leibniz Universität Hannover

Solution to exercise sheet 6

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Exercise 1:
A bipartite graph $G = (V, W, E)$ mit $||V|| = ||W||$ is regular if all vertices $v \in V \cup W$ have the same degree $\deg v \neq 0$. Show the following: Every regular bipartite graph has a perfect matching.

Solution: Let $G = (V, W, E)$ be a regular bipartite graph and the degree be $k \in \mathbb{N}$. Counting the edges we get $k \cdot ||V|| = k \cdot ||W||$. Hence it suffices to check Hall's condition because a matching which saturates V also saturates W and hence is a perfect matching. Let $S \subseteq V$ and let m_S be the number of edges from S to $\Gamma(S)$. As G is regular with degree k it holds that $m_S = k||S||$. These m_S edges are incident to $\Gamma(S)$ and hence $m_S \leq k||\Gamma(S)||$. Thus it holds $k||S|| \leq k||\Gamma(S)||$, whence $||\Gamma(S)|| \geq ||S||$ is true for $k > 0$ (which is true due to the definition of regular). As S has been chosen arbitrarily the result follows by Hall's theorem. □

Exercise 2: Let $G = (U, V, E)$ be a bipartite graph, where $U = \{1, \ldots, 4\}$ and $V = \{5, \ldots, 8\}$. The edges are defined as

$$E = \{\{1, 5\}, \{1, 6\}, \{2, 5\}, \{3, 6\}, \{3, 8\}, \{4, 7\}\}.$$

1. Is there a perfect matching for G? Use Hall's theorem in your argumentation.
2. Construct a network through adding source and sink and compute a maximum matching with Ford-Fulkerson.

Solution:

1. Yes, because for every $A \subseteq U$ it holds $||A|| \leq ||N(A)||$.
2. The maximum matching we get is a perfect one:

$$(1, 6), (2, 5), (3, 8), (4, 7).$$

□

Exercise 3: Compute a maximum matching of G in the previous task with the technique of augmenting paths. Use a BFS to find augmenting paths.

Solution: The following table shows the steps of BFS finding the augmenting paths and getting the matchings.

augmenting path	**matching**
$(1,5)$	$(1,5)$
$(2,5),(5,1),(1,6)$	$(2,5),(1,6)$
$(3,8)$	$(2,5),(1,6),(3,8)$
$(7,4)$	$(2,5),(1,6),(3,8),(7,4)$

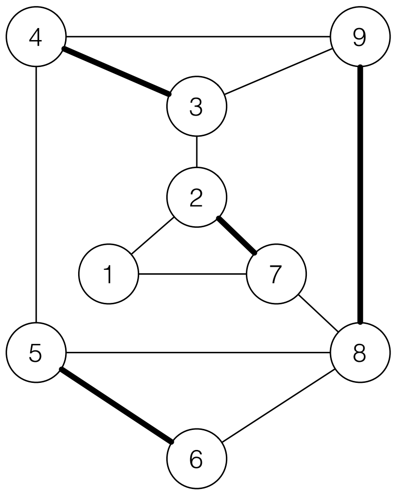
□

Exercise 4: Given graph $G = (V, E)$ where $V = \{1, \ldots, 9\}$ and

$$E = \{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{5,6\},\{7,8\},\{8,9\},\\ \{1,7\},\{2,7\},\{3,9\},\{4,9\},\{5,8\},\{6,8\}\}.$$

Compute a maximum matching of G with Edmonds algorithm.

Solution:



□