Efficient Algorithms
Summer term 2013

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Solution to exercise sheet 7

14.05.2013

Exercise 1: Generalize Euler's formula to unconnected plane graphs. Prove your claim.

Solution: Euler's formula can be generalized as follows. Let $G = (V, E)$ be a plane graph with c components. For each component Euler's formula $n - m + r = 2$ holds. Now we need to take care of the outer region. Every component shares this outer region. Hence we have $c-1$ regions less. It follows $n+r-(c-1) = m+2$ or $n+r-c-m = 1$. □

Exercise 2: Prove the following:
Let G be a triangle-free, connected, plane graph with $n(G) \geq 3$. Then it holds

$$m(G) \leq 2 \cdot n(G) - 4.$$

Solution: In a triangle-free graph which is not a forest any frontier of every region consists of at least 4 edges. Hence it holds $4 \cdot r(G) \leq 2 \cdot m(G)$. From Euler's formula we get

$$n(G) - m(G) + \frac{1}{2} m(G) \geq 2 \Longleftrightarrow m(G) \leq 2 \cdot n(G) - 4.$$

□

Exercise 3: Deduce from the previous task that $K_{3,3}$ is not planar.

Solution: In particular every bipartite graph is triangle-free. As $m(K_{3,3}) = 9 > 8 = 2 \cdot 6 - 4 = 2 \cdot n(K_{3,3}) - 4$ we get $K_{3,3}$ is not planar. □

Exercise 4: Construct an algorithm which colors any arbitrary graph G with at most $\Delta(G) + 1$ colors.

Solution:

Algorithm 1: Algorithm to color an undirected graph G with at most $\Delta(G)+1$ colors.

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Input : graph G = (V, E), some fixed vertex enumeration v_1, ..., v_n ∈ V(G)
for i = 1 to n do
    let c ∈ ℕ be the smallest number such that f(v) ≠ c for all {v_i, v} ∈ E;
    set f(v_i) := c.
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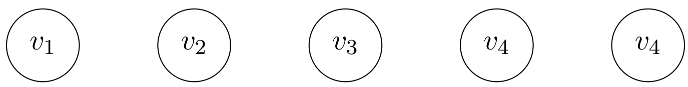
Claim. Let G be some graph, then $\chi(G) \leq \Delta(G) + 1$.

Proof of claim.
Suppose the vertices of the graph have an arbitrary labeling $1, \ldots, n$. W.l.o.g. suppose

the algorithm processes vertices in the order 1 to n. Further assume that the $\Delta(G)+1$ colors are $C := \{1, \ldots, \Delta(G)+1\}$. Now the induction hypothesis is that after i vertices have been processed, each of the vertices 1 to i is assigned a color in the set C such that for any edge $\{u, v\} \in E$ and $u, i \leq i$ the color $f(u) \neq f(v)$, where $f \colon V \to C$ is the constructed coloring in the algorithm.
Induction basis. After 1 vertex has been processed, color 1 is used for the first vertex and we are done. ✓
Induction step. $i \to i+1$: Vertex $i+1$ has at most $\Delta(G)$ neighbors. Hence it has at most $\Delta(G)$ neighbors that have already been assigned color. Thus there is at least one color in the set C that has not been used for any of $i+1$'s neighbors. The algorithm takes just the smallest remaining number in C to color vertex $i+1$. Hence the hypothesis still holds after assigning vertex $i+1$ a color. □

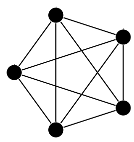
Exercise 5: Construct two examples of graphs G with $V(G) > 4$ such that the algorithm from the previous task needs $\Delta(G)+1$ colors and one where it requires less.

Solution: The following graph will be colored with exactly $1 = \Delta(G)+1$ color by the algorithm.



The following graph will be colored with exactly $5 = \Delta(G)+1$ colors by the algorithm.

K_5 :



□