Efficient Algorithms
Summer term 2013

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Solution to exercise sheet 5

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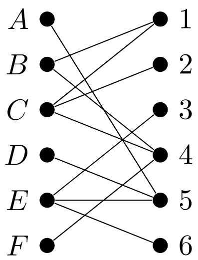
Exercise 1: Six reporters Arne (A), Barbara (B), Christine (C), Daniela (D), Elvis (E) and Frank (F), are to be assigned to six news stories Politics (1), Crime (2), Financial (3), Foreign (4), Local (5) and Sport (6). The table shows possible allocations of reporters to news stories. For example, Christine can be assigned to any one of stories 1, 2 or 4.

	1	2	3	4	5	6
A					✓	
B	✓			✓		
C	✓	✓		✓		
D					✓	
E			✓		✓	✓
F				✓		

1. Show these possible allocations on a bipartite graph.
2. Use Ford-Fulkerson to compute a maximum matching.
3. Is there a perfect matching? Explain your answer.

Solution:

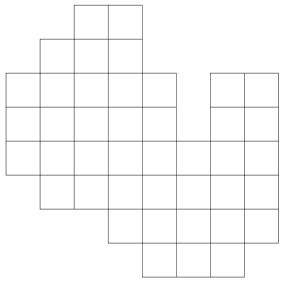
1.



2. $A = 5, B = 1, C = 2, E = 6, F = 4$.
3. Looking at the bipartite graph, there is only one possible pair for each of A and D, both of which can only be paired to 5. Since 5 cannot be paired with two different vertices, a complete matching is not possible, since one of A and D will always remain unpaired.

□

Exercise 2: Can the following figure be tiled by dominoes (a domino being 2 adjacent squares)? Give a tiling or a short proof that no tiling exists.



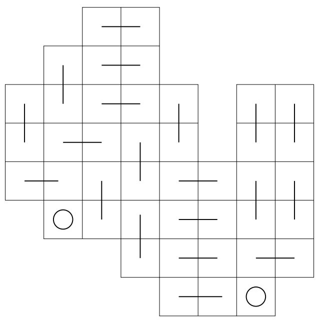
Solution: Consider the bipartite graph G with a vertex for each square and two squares are adjacent if they share an edge. This graph is bipartite since the squares can be colored black and white in a checkerboard pattern. Any perfect tiling gives a perfect matching by simply selecting the edges corresponding to the dominoes selected and vice versa.

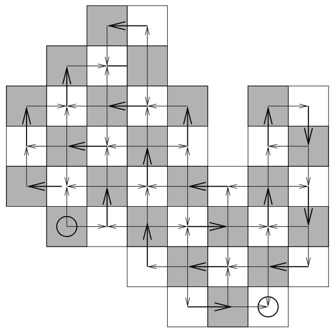
We claim that the configuration shown below is a maximum one and so no perfect tiling exists. We will prove that the matching M corresponding to the shown configuration is maximum by showing that there is no augmenting path as in the lecture. (Alternatively we could use Hall's theorem.)

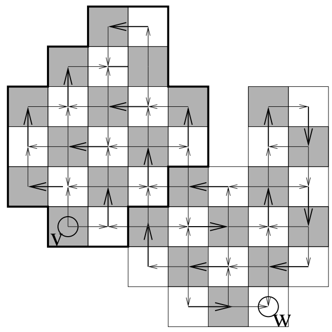
Let A be the set of black squares and B the set of white squares. Orient the edges of G according to M, i.e., all the edges in M are oriented from B to A, and the edges not in M are oriented from A to B as in the oriented version.

Let v be the only exposed vertex of A and w be the only exposed vertex of B, and consider L to be the set of vertices reachable from v (the enclosed area in the oriented version). Since w is not in L we obtain that no augmenting path exists.

We can also deduce the fact that no perfect matching exists from Hall's theorem by observing that the 11 black vertices in L (the enclosed region on the right) have only 10 (white) neighbors.







$\square$

Exercise 3: Consider a bipartite graph $G = (V, E)$ with bipartition $(A, B) : V = A \cup B$. Assume that, for some vertex sets $A_1 \subseteq A$ and $B_1 \subseteq B$, there exists a matching M_A covering all vertices in A_1 and a matching M_B covering all vertices in B_1. Prove that there always exists a matching covering all vertices in $A_1 \cup B_1$.

Solution: Let $G = (V, E) = (A \cup B, E)$, subsets $A_1 \subset A, B_1 \subset B$ and matchings M_A, M_B that cover A_1 and B_1, respectively. We construct a matching M that covers $A_1 \cup B_1$. Clearly, the edge set $M = M_A \cup M_B$ covers $A_1 \cup B_1$, but it is not necessarily a matching. We show how to delete edges from M to make it into a matching. We know $M_A \Delta M_B$ is a union of disjoint cycles and alternating paths. The vertices with some incident edge from both $M_A \setminus M_B$ and from $M_B \setminus M_A$ are the only ones where M fails to be a matching. We show how to delete some edges from $M_A \Delta M_B$, so M is still a matching and no vertices are uncovered. We do so in each component of $M_A \Delta M_B$.

Cycle: Since G is bipartite, the cycle has even length. Therefore, we can delete every other edge and the desired properties hold.

Path of odd number of edges: We can delete every other edge starting from the edge that is adjacent to the last edge of the path. The desired properties hold. Note that this is possible only because the path has odd number of edges.

Path of even number of edges: In this case, we can delete every other edge but one endpoint will be covered and the other uncovered. We need to prove that both endpoints cannot be in $A_1 \cup B_1$. Thus, we delete every other edge so the endpoint that is not in $A_1 \cup B_1$ is uncovered.

We do so by contradiction: assume both endpoints are in $A_1 \cup B_1$. As the path has an even number of edges, and G is bipartite, then both endpoints must belong to the same bipartition set (A or B). W.l.o.g. say they both belong to A, and thus also belong to A_1. Note that each vertex in A_1 has exactly one incident edge from M_A; thus the path we are analyzing (that is a connected component of $M_A \Delta M_B$) must contain these two edges. However, this path is of even length, and is alternating, so the end-edges cannot be from the same matching M_A ($\Rightarrow\Leftarrow$). This shows that our initial assumption is wrong, i.e., it must happen that both endpoints do not belong to $A_1 \cup B_1$, as desired.