Efficient Algorithms
Summer term 2013

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Solution to exercise sheet 10

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Another possibility for parallel computers is to consider the message passing model where a global storage is not used. Formally one sees processors as vertices in a undirected graph and the edges are bidirectional communication channels. Still, the processors work synchronously with the following exceptions: After execution of the operation

- `send(Object, ProcessorId)` the sender-processor can immediately continue,
- `receive(Variable, ProcessorId)` the receiver-processor waits until he gets the desired object.

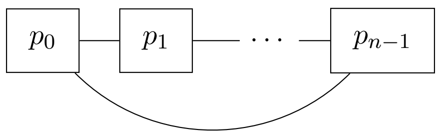
The underlying net-topology has three important attributes:

Diameter Maximum distance of two vertices.

Degree Maximum degree of a vertex.

Bisection width Divide the set of vertices into two almost equal halves. The bisection width is then the number of edges connecting both sets.

Exercise 1: A *processor ring* is an array consisting of n processors which are linearly linked and the last processor in the array is linked to the first:



1. What are the attribute values of this topology?
2. Given an $n \times n$ matrix A and an n-vector $\vec{b}$ one shall compute $A \cdot \vec{b}$. Construct an algorithm on a processor ring with $N \leq n$ processors running in $\Theta(\frac{n^2}{N})$ time with communication time $\Theta(N \cdot n)$.

Solution:

1. **Diameter** $\left\lfloor \frac{N}{2} \right\rfloor$

   Degree 2

   Bisection width 2

2. W.l.o.g. let N be a divisor of n and let $r = \frac{n}{N}$. Now the idea is to divide A into N sub matrices of size $n \times r$,

$$A = \left(\underbrace{A_0}_{r} \;\vdots\; \underbrace{A_1}_{r} \;\vdots\; \underbrace{\cdots}_{r} \;\vdots\; \underbrace{A_{N-1}}_{r} \right)$$

and $\vec{b}$ such that b_i is a tuple of length r. Then it holds that

$$A \cdot \vec{b} = A_0 \cdot b_0 + A_1 \cdot b_1 + \cdots + A_{N-1} \cdot b_{N-1}.$$

Algorithm 1: matMultVec $\langle$Ring$\rangle$

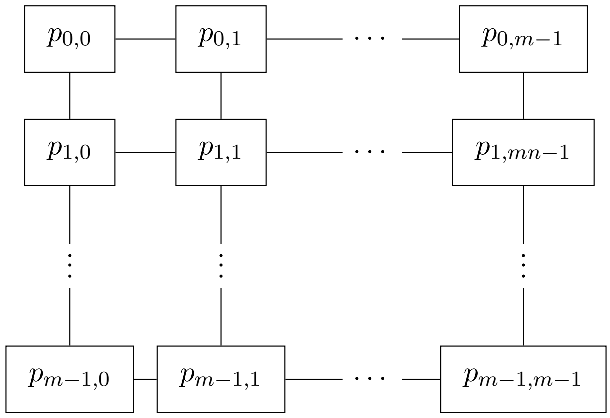
N, p: integer; // initialized $0, \leq p < N$
B: array$[1,\dots,n][1,\dots,r]$ of real; // Submatrix A_p
w: array$[1,\dots,r]$ of real; // Subvector b_p
y, z: array$[1,\dots,n]$ of real;
Compute locally $z = A_p \cdot b_p$;
if $p = 0$ **then** $y := 0$;
else receive$(y, p-1)$;
$y := y + z$;
send$(y, (p+1) \mod N)$;
if $p = 0$ **then** receive$(y, N-1)$;

Observations:

- The processors compute in parallel the products $A_p \cdot b_p$. Then all processors with ids 1 to $N-1$ wait for the data of their left neighbor. P_0 sets y to $A_0 \cdot b_0$ and sends the result to the right neighbor etc. P_0 then waits for the full sum.
- The computation time is $\Theta(r \cdot n) = \Theta(\frac{n^2}{N})$.
- The communication time is $\Theta(N \cdot n)$ because P_0 has to wait until all values are summed up.

$\square$

Exercise 2: A *processor grid* consists of $N = m^2$ processors which are arranged as an $m \times m$ matrix and are connected as follows:



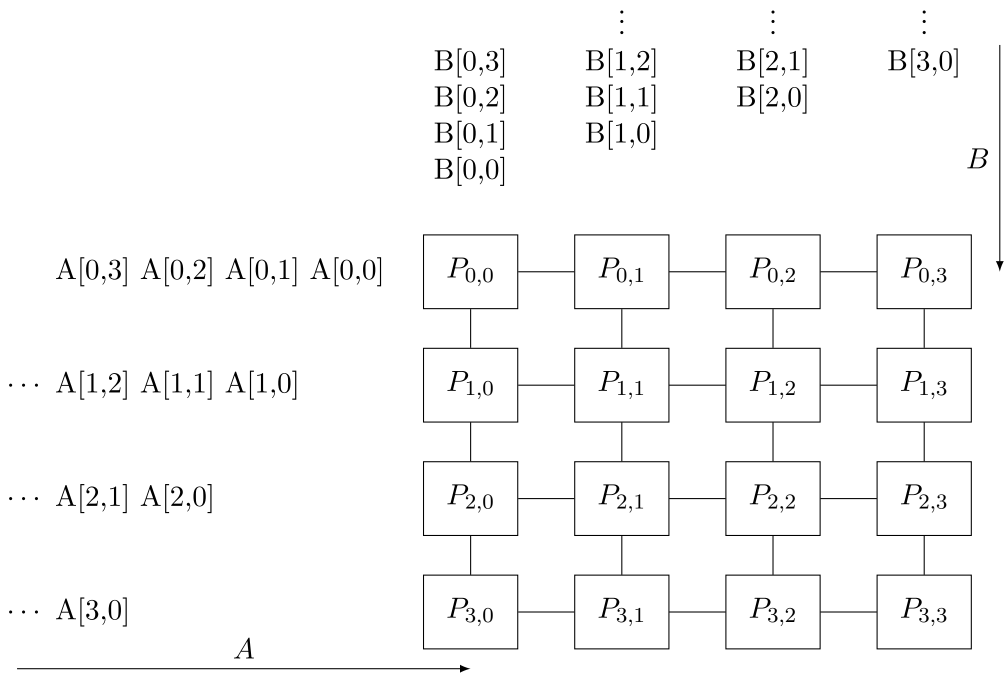
1. What are the attribute values of this topology?
2. Construct an algorithm computing the product of two $n \times n$ matrices in $\Theta(n)$ steps.

Solution:

1. **Diameter** $2(m-1)$

   Degree 4

   Bisection width m
2. **Idea:** The rows of A will be entered in inverted order from left the columns of B from top into the grid. Every processor P_{ij} gets in every step two inputs $A[i,\ell]$ and $B[\ell,j]$, sums the product and sends $A[i,\ell]$ to the right and $B[\ell,j]$ to bottom.



Algorithm 2: matmul $\langle$Grid$\rangle$

n, i, j: integer; // initialized, $(i, j) =$ processor id
A, B, C: real; // A contains $A[i,k]$ and B contains $B[k,j]$
d: integer;
$C := A \cdot B$;
for $d := 1$ *to* $n - 1$ **do**
  send$(A, (i, (j+1) \mod n))$;
  send$(B, ((i+1) \mod n, j))$;
  receive$(A, (i, (j-1) \mod n))$;
  receive$(B, ((i-1) \mod n, j))$;
  $C := C + A \cdot B$;

Observations:

- After $\Theta(n)$ steps processor P_{ij} has computed the correct value of $C[i,j]$.
- Systolic algorithm. The values flow in common mode through the grid.
- After $\Theta(n)$ steps matrix C can be read.
- Speedup of magnitude n^2 — hence optimal speedup.
- The coefficient of A and B can be divided onto the processors of a torus and use the same algorithm.

□