Advanced Logics
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Solution to exercise sheet 1

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Exercise 1: Which of the following claims is correct, which is wrong, and why?

1. For some $c \in \{\top, \bot\}$ it holds $c \in D$.
2. The function *nand* $x|y$ is functional complete.
3. $(x \wedge y) \vee (x \wedge z) \vee (y \wedge z) \in \mathrm{PL}(D_1)$.
4. For every $n \geqslant 4$ it holds $T_2^{n+1} \in [T_2^n]$.
5. $x \rightarrow y$ is representable in $\{x \vee y, x \leftrightarrow y\}$.
6. Every 0-separating function of degree 1 is 1-reproducing.

Solution:

1. False, constants are not self-dual.
2. True: $\neg x \equiv x|x$, $x \wedge y \equiv (x|y)|(x|y)$, $x \vee y \equiv (x|x)|(y|y)$.
3. True: $(x \wedge y) \vee (x \wedge z) \vee (y \wedge z) \equiv \mathbf{maj}\{x, y, z\} \in D_2 \subset D_1$.
4. True: $T_2^{n+1}(x, \ldots, x, y, z) \equiv x \vee (y \wedge z)$. Hence: $T_2^4(x, x, y, z) = x \vee (y \wedge z) =: f(x, y, z)$. Thus:

$$\begin{aligned} T_2^{n+1}(x_1, \ldots, x_{n+1}) &= T_2^n(x_1, \ldots, x_n) \vee \bigvee_{i=1}^{n} (x_i \wedge x_{n+1}) \\ &= f(f(\cdots f(f(T_2^n(x_1, \ldots, x_n), x_1, x_{n+1}), x_2, x_{n+1}) \cdots), x_n, x_{n+1}) \end{aligned}$$

5. True:

$$\begin{aligned} y \leftrightarrow (x \vee y) &= \big(y \rightarrow (x \vee y)\big) \wedge \big((x \vee y) \rightarrow y\big) \\ &= \big(\neg y \vee x \vee y\big) \wedge \big((\neg x \wedge \neg y) \vee y\big) = (\neg x \vee y) \wedge (y \vee \neg y) = x \rightarrow y \end{aligned}$$

6. True. We prove this by contradiction. Let f be 0-separating of degree 1. Hence f.a. $A \subseteq f^{-1}(0)$ with $|A| = 1$ this tuple in A contains a 0. Assume f is not 1-reproducing. Thus $f(1, \ldots, 1) = 0$. In particular also for $\{(1, \ldots, 1)\} \subseteq f^{-1}(0)$ the set $\{(1, \ldots, 1)\}$ has to be 0-separating. Obviously not. Contradiction.

$\square$

Exercise 2: Construct a function f, which is 0-separating of degree 4 and show that f can be represented in $\{x \to y, T^5_2\}$.

Solution: Search a functions $f(x_1, \dots, x_5)$ s.t. all $A \subseteq f^{-1}(0)$ with $|A| = 4$ are 0-sep. Truth table for such an f is:

x_1	x_2	x_3	x_4	x_5	f
0	0	0	0	1	0
0	0	0	1	0	0
0	0	1	0	0	0
0	1	0	0	0	0
1	0	0	0	0	0
a_1	a_2	a_3	a_4	a_5	1

for all remaining a_is. Thus: f neither monotone nor 0-reproducing, as $f(0, \dots, 0) = 1$. Thus it must be in $S^4_0 = [x \to y, T^5_2]$. At least two arguments have to be 1 such that $f(\bar{x}) = 1$ can hold. Use $T^5_2(x_1, \dots, x_5)$ to express this unless $x_1 = \dots = x_5 = 0$ holds. Then we have $f(\bar{}) = 1$ and thus we have

$$f(x_1, \dots, x_5) \equiv (x_1 \vee \dots \vee x_5) \to T^5_2(x_1, \dots, x_5).$$

Of course we can express disjunctions of arbitrary arity, as we have a binary disjunction: $x \vee y \equiv T^5_2(x, x, x, y, y)$. $\square$

Exercise 3: For any of the following formulas $\varphi_1, \dots, \varphi_6$ determine a minimal[1] set B of Boolean functions s.t. $\varphi_i \in \mathrm{PL}(B)$ holds and further state if φ_i is in an tractable or intractable fragment of $\mathrm{SAT}(B)$ with the help of Lewis' dichotomy theorem:

$$\varphi_1 := (x \wedge y) \vee (z \wedge \top), \qquad \varphi_2 := (x \not\to y) \vee (z \wedge w), \qquad \varphi_3 := (x \vee z) \wedge \big(x \to y\big),$$
$$\varphi_4 := (x \to y) \vee ((\bot \to x) \wedge y), \qquad \varphi_5 := (x \not\to \neg y), \qquad \varphi_6 := (x \oplus y) \wedge (z \vee x \vee \bot).$$

Solution: Es gilt:

$$\begin{aligned}
&\varphi_1 \equiv (x \wedge y) \vee z \in [(x \wedge y) \vee z] = S_{00} && \text{(efficient)},\\
&\varphi_2 \in [\{(x \not\to y) \vee z\} \cup \{\wedge\}] = [S_{02} \cup E_2] = R_2 && \text{(efficient)},\\
&\varphi_3 \in [\{(x \to y) \wedge z\} \cup \{\vee\}] = [S_{12} \cup V_2] = R_2 && \text{(efficient)},\\
&\varphi_4 \equiv (x \to y) \vee y = x \to y \in [\to] = S_0 && \text{(efficient)},\\
&\varphi_5 \equiv x \wedge y \in [\{\wedge\}] = E_2 && \text{(efficient)},\\
&\varphi_6 \equiv (x \oplus y) \wedge (z \vee x) \in [\{x \oplus y\} \cup \{x \wedge (y \vee z)\}] = L_0 \cup S_{10} = R_0. && \text{(inefficient)}.
\end{aligned}$$

$\square$

[1] We say that B is minimal w.r.t. $\varphi \in \mathrm{PL}(B)$ iff $[B] \subseteq \mathsf{BF}$ and there exists no B' with $[B'] \subset [B]$, such that also $\phi \in \mathrm{PL}(B')$ holds.

Exercise 4: Prove that every monotone function $f(x_1, \ldots, x_n)$ can be represented in $\{0, 1, \wedge, \vee\}$.

Solution: Every monotone function can be represented by the following DNF:

$$f_{\text{mDNF}} := \bigvee_{\substack{x_1,\ldots,x_n \in \{0,1\},\\ f(x_1,\ldots,x_n)=1}} \bigwedge_{\substack{1 \leqslant i \leqslant n,\\ x_i = 1}} x_i,$$

where empty conjunctions are true and empty disjunctions are false. Correctness immediately follows from definition of monotonicity. □

Exercise 5: Prove that the set $\{0, 1, \wedge, \vee\}$ is functional maximal incomplete (without using arguments on the structure of Post's lattice).

Solution: Assume the opposite: $S := \{0, 1, \wedge, \vee\}$ is not maximal functional incomplete. Two cases:

1. S is functional complete. We claim *only* monotone functions can be represented by S. By this it follows that $\neg \notin [S]$.

   Proof. Obviously $0, 1, \wedge$, and $\vee$ are monotone. Use induction to prove that compositions are still monotone. Assume $f(\vec{x})$ and $g(\vec{y})$ are monotone. Then $\phi := f(\vec{x}) \wedge g(\vec{y})$ is still monotone because of the following. Let θ be an assignment over $\vec{x}, \vec{y}$. If $\theta \models f(\vec{x}) \wedge g(\vec{y})$ then $\theta|_{\vec{x}} \models f(\vec{x})$ and $\theta|_{\vec{y}} \models g(\vec{y})$. By IH for any assignment $\xi \geqslant \theta$ we get $\xi|_{\vec{x}} \models f(\vec{x})$ and $\xi|_{\vec{y}} \models g(\vec{y})$, whence $\xi \models f(\vec{x}) \wedge g(\vec{y})$. Thus $f \wedge g$ is monotone.

   Now the same for $f(\vec{x}) \vee g(\vec{x})$. Let θ as above. If $\theta \models f(\vec{x}) \vee g(\vec{y})$ then $\theta|_{\vec{x}} \models f(\vec{x})$ or $\theta|_{\vec{y}} \models g(\vec{y})$ (at least one of them!). W.l.o.g. $\theta|_{\vec{x}} \models f(\vec{x})$. Then, by IH, for any assignment $\xi \geqslant \theta$ it holds $\xi|_{\vec{x}} \models f(\vec{x})$ and thus $\xi \models f(\vec{x}) \vee g(\vec{y})$ follows. Hence $f \vee g$ is monotone.

2. S is not functional maximal incomplete and not functional complete. Thus we have function f which cannot be represented in $\{0, 1, \wedge, \vee\}$ and also $\{0, 1, \wedge, \vee, f\}$ is *not* functional complete. Let $f(x_1, \ldots, x_n)$ be such a function and $n \in \mathbb{N}$. As f is not monotone (all monotone functions can be represented in $\{0, 1, \wedge, \vee\}$) there is at least one assignment $\bar{a} \in \{0, 1\}^n$, s.t. $f(\bar{a}) = 1$ holds and another assignment $\bar{b} \in \{0, 1\}^n$ with $\bar{b} > \bar{a}$ s.t. $f(\bar{b}) = 0$ (otherwise f is monotone). It holds $a_i \leqslant b_i$ and, in particular, there is at least one $1 \leqslant j \leqslant n$ s.t. $a_j < b_j$. W.l.o.g. $\bar{b}$ is minimal, i.e., $a_i = b_i$ for $1 \leqslant i \neq j \leqslant n$.

   Hence $f(a_1, \ldots, a_{j-1}, x, a_{j+1}, \ldots, a_n) = \neg x$ contradicting the assumption that $S \cup \{f\}$ is not functional complete.

Hence $\{0, 1, \wedge, \vee\}$ is functional maximal incomplete. □

Exercise 6: Given $f \in B_3$ via its truth-table:

x_1	x_2	x_3	f
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

Determine the minimal clone $\mathcal{B}$ s.t. $f \in \mathcal{B}$ holds and explain your claim..

Solution: What can you read out of the truth table:

- f is 1-reproducing: $f(1,1,1) = 1$.
- f is not 0-reproducing: $f(0,0,0) = 1$.
- f is not monotone: $f(0,0,1) = 1$ and $f(0,1,1) = 0$.
- f is not self-dual: $f(0,0,0) = 1 \neq 0 = \neg f(1,1,1)$. (already follows from previous cases)
- f is not unary: $f \notin N$. (already followed)
- f is not affine (f cannot be written as $c_0 \oplus c_1x_1 \oplus c_2x_2 \oplus c_3x_3$):
  $f(0,0,0) = 1$ implies $c_0 = 1$,
  $f(0,0,1) = 1$ implies $c_3 = 0$, as $c_0 = 1$,
  $f(0,1,0) = 1$ implies $c_2 = 0$, as $c_0 = 1$, a
  $f(1,0,0) = 1$ implies $c_1 = 0$, as $c_0 = 1$. Together $f \equiv 1$ which is not true.)
- f is neither 0-separating nor 0-separating of degree 2: $f(0,1,1) = 0$ requires $x_1 = 0$, but $f(1,1,0) = 0$.

Hence R_1 is the minimal clone of f. $\square$