Advanced Logics
Winter term 2015

Institut für Theoretische Informatik
Leibniz Universität Hannover

Solution to exercise sheet 5

Exercise 1: Given the hybrid logic formula $\phi :=\downarrow x.(\phi_S \wedge \phi_C \wedge \phi_I \wedge \phi_D \wedge \phi_4)$, where

$$\begin{aligned}
\phi_S &:= x \wedge \neg\Diamond x \wedge \Diamond\neg x \wedge \Box\Diamond x,\\
\phi_C &:= \Box\Box \downarrow y.(\neg x \rightarrow \Diamond(x \wedge \Diamond y)),\\
\phi_I &:= \Box \downarrow y.\neg\Diamond y,\\
\phi_D &:= \Box\Diamond\neg x,\\
\phi_4 &:= \Box \downarrow y.\Diamond(x \wedge \Box((\Diamond(\neg x \wedge \Diamond y)) \rightarrow \Diamond y)).
\end{aligned}$$

Prove that every model for ϕ is infinite.

Solution: We follow

> Blackburn, P. (2000). *Representation, Reasoning, and Relational Structures: a Hybrid Logic Manifesto.* Logic Journal of the IGPL, Vol. 8 No. 3, pp. 339–365, Oxford University Press 2000.

Let $(\mathbb{N}, <)$ be the frame of natural numbers with the usual ordering and $s \notin \mathbb{N}$ (s is our spypoint). Let $\mathcal{N}^s$ be the following model

$$\begin{aligned}
&W := \mathbb{N} \cup \{s\},\\
&R := < \cup \{(n,s),(s,n) \mid n \in \mathbb{N}\},\\
&V \text{ arbitrary.}
\end{aligned}$$

Obviously it holds $\mathcal{N}^s, s \models \phi$, as:

- S: s is not reflexive, there exists a reachable successor different from s, and from every $<$-successor of s there is an R-edge back to s,
- C: if we are in a non-x-state y then we can reach x and go back to y,
- I: every number is non reflexive,
- D: from every number we can go to a successor w.r.t. $<$,
- 4: R is transitive.

Of course we have $|\mathcal{N}^s| = \infty$. More, every model $\mathcal{M} = ((W,R),V)$ of ϕ is of infinite size. Choose some $\mathcal{M}, s \models \phi$. Let $B = \{b \in W \mid sRb\}$ the set of reachable worlds from s.
As ϕ_S holds we get $s \notin B$, $B \neq \emptyset$, and for all $b \in B$ it holds that $(b,s) \in R$.

As ϕ_C holds we get from $a \neq s$ and a is a R-successor of one world from B that $a \in B$.
As ϕ_I holds any world in B is irreflexive.
As ϕ_D holds every world in B has an R-successor different from s.
As ϕ_4 holds R is transitive on B.
Thus B is an unrestricted strict partial order whence B is of infinite size and consequently W, too.
Note: We have seen that the ability to bind information locally enables us to express strong properties leading to large structures of infinite size. This power is one reason for the undecidability of the satisfiability problem in this logic.

$\square$