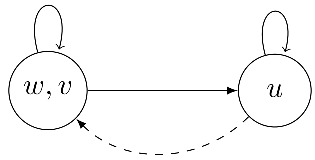
Solution to exercise sheet 3

Exercise 1: A frame (W, R) is said to be *symmetric* iff $\forall u, v \in W\colon uRv \to vRu$. Prove that every **S**5-frame is symmetric

Solution: Let (W, R) be an **S**5-frame. Hence we have: $u, v, w \in W$ with $w = v$ and $w \neq u, w \neq v$ implies wRv due to reflexivity. If wRu then uRv as $\forall u, v, w \in W$: $wRu \wedge wRv \to uRv$.



$\square$

Exercise 2: Prove that **S**5-ML-SAT$_{\square,\Diamond}$(BF) is NP-complete w.r.t. $\leq_{\mathbf{m}}^{\mathbf{log}}$ reductions.

Solution: We follow [Ladner 1977, p. 479]. First we show a *short model property* for this kind of formulas.

Claim. *If* $\phi \in$ **S**5-ML-SAT$_{\square,\Diamond}$(BF) *and* $m = |\phi|_\square + |\phi|_\Diamond$ *then* ϕ *can be satisfied in a* **S**5*-frame with* $\leq m + 1$ *worlds.*

Proof of Claim. Let $\phi \in$ **S**5-ML-SAT$_{\square,\Diamond}$(BF) via (reflexive, transitive, symmetric, euclidean) **S**5-model $\mathcal{M} = ((W, R), V)$. Hence (W, R) is a complete graph. Construct mapping $\tau\colon \mathbf{SF}(\phi) \to \mathcal{W}$ s.t. ϕ is satisfied in **S**5-model $(\text{Range}(\tau), R|_{\text{Range}(\tau)}, V|_{\text{Range}(\tau)})$ and $|\text{Range}(\tau)| \leq m + 1$. τ is inductively defined:

$$\text{set } \tau(\phi) \in W, \text{ s.t. } \tau(\phi) \in \tilde{V}(\phi), \tag{1}$$

$$\tau(\beta) = \tau(\alpha) \text{ if } \alpha = \neg\beta, \quad \text{(eval. prop. formulas in same world)} \tag{2}$$

$$\tau(\beta) = \tau(\gamma) = \tau(\alpha) \text{ if } \alpha = \beta \wedge \gamma, \quad (\text{——„——}) \tag{3}$$

$$\text{if } \alpha = \square\beta \text{ and } \tau(\alpha) \notin \tilde{V}(\alpha)\text{, then choose } \tau(\beta) \in W\text{, s.t. } \tau(\beta) \notin \tilde{V}(\beta), \tag{4}$$

$$(\Diamond \text{ moves to new worlds})$$

where $\alpha \in \mathbf{SF}(\phi)$ and $\tilde{V} : \mathbf{SF}(\text{ML}) \to \mathcal{P}(W)$ is the valuation V extended on modal formulas as follows

$$\begin{aligned} w \in \tilde{V}(\alpha \wedge \beta) &\text{ iff } w \in \tilde{V}(\alpha) \text{ and } w \in \tilde{V}(\beta), \\ w \in \tilde{V}(\neg\alpha) &\text{ iff } w \notin \tilde{V}(\alpha), \\ w \in \tilde{V}(\square\alpha) &\text{ iff } \forall w' \in W : wRw' \to w' \in \tilde{V}(\alpha). \end{aligned}$$

By construction: $|\mathrm{Range}(\tau) \leq m+1$. Set $\mathcal{M}' = ((W', R'), V')$ with $W' = \mathrm{Range}(\tau)$, $R' = R|_{W'}$, and $V' = V|_{W'}$. For all $\psi \in \mathbf{SF}(\phi)$ we have $\tau(\psi) \in \tilde{V}(\psi)$ iff $\tau(\psi) \in \tilde{V}'(\psi)$ (by induction). $\dashv$

The NP-algorithm just guesses such a model of size linear in $|\phi|$ ans afterwards verifies it via the polynomial time model checking algorithm.
As SAT(BF) is a special case of $\mathbf{S}5$-ML-SAT$_{\Box,\Diamond}(\mathsf{BF})$ we get NP-hardness and together NP-completeness. $\Box$