Solution to exercise sheet 4

Exercise 1: Consider the reduction in the proof of Theorem 2.12, which shows coNP-hardness of Poor Man's Conjunctive Satisfiability. The reduction is constructed from

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Input: Formula $\phi = \bigwedge_{i=1}^{m} \bigvee_{j=1}^{3} l_{i,j}$ and $l_{i,j} \in \{x_1, \dots, x_n\}$

Question: Is there a *traversal*, hence an assignment $\theta\colon \{x_1, \dots, x_n\} \to \{0, 1\}$, s.t. f.a. $1 \leq i \leq m$ there is only one $1 \leq j \leq 3$ with $\hat{\theta}(l_{i,j}) = 1$?

The reduction $f(\phi)$ then was

$$f(\phi) = (\mathcal{X}_1)^2 \top \wedge \cdots \wedge (\mathcal{X}_n)^2 \top \wedge \Box^{2m} \bot$$

$$\mathcal{X}_i = \Delta_i^1 \cdots \Delta_i^m, \quad \Delta_i^j = \begin{cases} \Diamond & \text{, if } x_i \in C_j \\ \Box & \text{, otherwise} \end{cases} \text{ for } 1 \leq i \leq n$$

1. Compute the value of

$$\phi = (x_1 \vee x_3 \vee x_5) \wedge (x_2 \vee x_3 \vee x_4) \wedge (x_2 \vee x_4 \vee x_5)$$

and show that ϕ has a traversal. Further show that $f(\phi)$ is unsatisfiable.

2. Compute the value of

$$\psi = (x_1 \vee x_2 \vee x_4) \wedge (x_2 \vee x_4 \vee x_5) \wedge (x_1 \vee x_3 \vee x_5) \wedge (x_2 \vee x_3 \vee x_4)$$

and show that there is no traversal for ψ. Further show that there exists a model $\mathcal{M}$ with $\mathcal{M} \models f(\psi)$.

Solution:

1. The value of $f(\phi)$ is

$$\Diamond\Box\Box\Diamond\Box\Box\top \wedge \Box\Diamond\Diamond\Box\Diamond\Diamond\top \wedge \Diamond\Diamond\Box\Diamond\Diamond\Box\top \wedge \Box\Diamond\Diamond\Box\Diamond\Diamond\top \wedge \Diamond\Box\Diamond\Diamond\Box\Diamond\top \wedge \Box^6\bot.$$

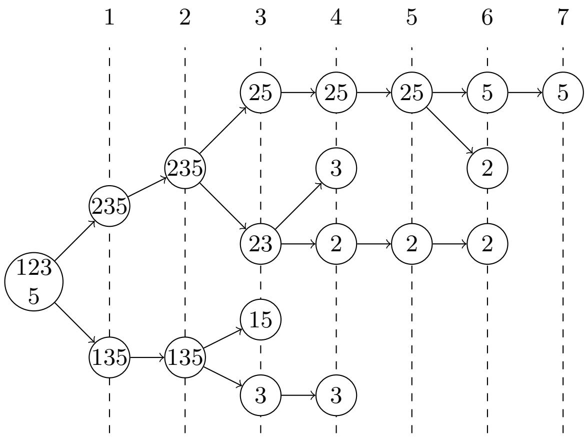
Of course ϕ has a traversal: $\theta(x_1) = \theta(x_2) = 1$ and $\theta(x_3) = \theta(x_4) = \theta(x_5) = 0$. The only possible way to satisfy such a formula is to build in dead ends for satisfying the last conjunct. However the first two conjuncts prohibit this hence $f(\phi)$ is unsatisfiable.

The unsatisfiability can also be shown by an application of the `World` algorithm from the lecture.

2. The resulting formula $f(\phi)$ is

$$\underbrace{\Diamond\Box\Diamond\Box\Diamond\Box\Diamond\Box\top}_{\alpha_1} \wedge \underbrace{\Diamond\Diamond\Box\Diamond\Diamond\Diamond\Box\Diamond\top}_{\alpha_2} \wedge$$
$$\underbrace{\Box\Box\Diamond\Diamond\Box\Box\Diamond\Diamond\top}_{\alpha_3} \wedge \underbrace{\Diamond\Diamond\Box\Diamond\Diamond\Diamond\Box\Diamond\top}_{\alpha_4} \wedge$$
$$\underbrace{\Box\Diamond\Diamond\Box\Box\Diamond\Diamond\Box\top}_{\alpha_5} \wedge \Box^8\bot,$$

and it is satisfied by the following model. In the following we mark the relevant subtrees for satisfying the corresponding formula. Observe that $\alpha_2 = \alpha_4$.



$\Box^8\bot$ is everywhere in the model true.

$\Box$