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Solution to exercise sheet 8

Exercise 1: Given a set of propositional formulas Γ, and a propositional formula φ. The *implication problem* IMP is then the question if $\Gamma \models \varphi$ holds. Show that this problem is coNP-complete.

Solution: It holds that

$$(\Gamma, \varphi) \in \text{IMP} \quad \text{iff} \quad (\bigwedge_{\gamma\in\Gamma} \gamma) \to \varphi \in \text{TAUT}$$

which shows the upper bound. Further we have

$$\varphi \in \text{TAUT} \quad \text{iff} \quad (\{\top\}, \varphi) \in \text{IMP}$$

which proves the lower bound. $\square$

Exercise 2: Let B be a finite set of Boolean functions such that $\mathsf{S}_{11} \subseteq [B] \subseteq \mathsf{M}$. Then $\text{EXT}(B)$ is P^{NP}-complete w.r.t. $\leq^{\mathbf{p}}_{\mathbf{m}}$.
To show P^{NP}-hardness reduce from the following P^{NP}-complete problem:

Problem: SNSAT — sequentially nested satisfiability

Given: A sequence $(\varphi^i)_{1\leq i\leq n}$ of formulae such that φ^i contains the propositions $x_1, \ldots, x_{i-1}$ and $z_{i1}, \ldots, z_{im_i}$

Question: Is $c_n = \top$, where c_i is recursively defined via $c_i := \top$ if and only if φ^i is satisfiable by an assignment σ such that $\sigma(x_j) = c_j$ for all $1 \leq j < i$?

Hint: for the membership result you may use partially your result from Exercise 1.

Solution: We start by showing $\text{EXT}(B) \in \mathsf{P}^{\mathsf{NP}}$. Let B be a finite set of Boolean functions such that $[B] \subseteq \mathsf{M}$ and $\langle W, D\rangle$ be a B-default theory. As the negated justification $\neg\beta$ of every default rule $\frac{\alpha:\beta}{\gamma} \in D$ is either equivalent to the constant $\top$ or not $\top$-reproducing, it holds that in the former case $\neg\beta$ is contained in any stable extension, whereas in the latter $\neg\beta$ cannot be contained in a consistent stable extension of $\langle W, D\rangle$. We can distinguish between those two cases in polynomial time. Therefore, using the stage construction characterisation known from the lecture, we can iteratively compute the applicable defaults and test whether the premise of any default with unsatisfiable conclusion can be derived.
The algorithm implements these steps on a deterministic Turing machine using a coNP-oracle to test for implication of B-formulae which follows from Exercise 1. Clearly, the

Input: $\langle W, D\rangle$

$G_{\text{new}} \leftarrow W$;
repeat
  $G_{\text{old}} \leftarrow G_{\text{new}}$;
  forall the $\frac{\alpha:\beta}{\gamma} \in D$ **do**
    if $G_{\text{old}} \models \alpha$ ***and*** $\beta \not\equiv \bot$ **then**
      if $\gamma \equiv \bot$ **then return** false ;
      $G_{\text{new}} \leftarrow G_{\text{new}} \cup \{\gamma\}$;
until $G_{\text{new}} = G_{\text{old}}$;
return true

algorithm terminates after a polynomial number of steps. Hence, $\mathrm{EXT}(B)$ is contained in P^{NP}.
To show the P^{NP}-hardness of $\mathrm{EXT}(B)$, we show the reduction from SNSAT.
Essentially: instance of SNSAT sequence of formulae s.t. satisfiability of ith formula depends on the satisfiability of all of its predecessors.
Idea: encode this into sequence of default rules, i.e., resulting default theory has stable extension iff last formula is satisfiable ($c_n = 1$).
Start with $(\varphi^i)_{1\leq i\leq n}$ plus w.l.o.g. φ^i is in CNF f.a. $1 \leq i \leq n$. Introduce for x_j or z_{ij} occurring in $(\varphi^i)_{1\leq i\leq n}$ introduce fresh x'_j, z'_{ij} and define

$$\psi^i := \varphi^i[\neg x_1/x'_1, \ldots, \neg x_{i-1}/x'_{i-1}, \neg z_{i1}/z'_{i1}, \ldots, \neg z_{im_i}/z'_{im_i}] \wedge \bigwedge_{j=1}^{i-1}(x_j \vee x'_j) \wedge \bigwedge_{j=1}^{m_i}(z_{ij} \vee z'_{ij}).$$

Key observation: it holds f.a. $c_1, \ldots, c_{i-1} \in \{\bot, \top\}$,

$$\varphi^i[x_1/c_1, \ldots, x_{i-1}/c_{i-1}]$$

is unsatisfiable iff f.a. models σ of

$$\psi^i[x_1/c_1, \ldots, x_{i-1}/c_{i-1}, x'_1/\neg c_1, \ldots, x'_{i-1}/\neg c_{i-1}]$$

there exists an index $1 \leq j \leq m_i$ s.t. σ sets both z_{ij} and z'_{ij} to $\top$.
Idea: use this to show that $\langle W, D\rangle$ from below has a stable extension iff $(\varphi^i)_{1\leq i\leq n} \in$ SNSAT, i.e., $\varphi^n[x_1/c_1, \ldots, x_{i-1}/c_{i-1}]$ is satisfiable for $c_1, \ldots, c_{i-1}$ recursively defined via

$$c_i := \top \qquad \text{iff} \qquad \varphi^i[x_1/c_1, \ldots, x_{i-1}/c_{i-1}] \text{ is satisfiable.} \tag{1}$$

Define $W := \{\psi^1, \ldots, \psi^n\}$ and

$$D := \left\{ \frac{\bigvee_{j=1}^{m_i}(z_{ij} \wedge z'_{ij}) \vee \bigvee_{j=1}^{i-1}(x_j \wedge x'_j) : \top}{x'_i} \;\middle|\; 1 \leq i < n \right\} \cup \left\{ \frac{\bigvee_{j=1}^{m_n}(z_{nj} \wedge z'_{nj}) \vee \bigvee_{j=1}^{n-1}(x_j \wedge x'_j) : \top}{\bot} \right\}.$$

Start with $E_0 := W$. If φ^1 is unsatisfiable then $\frac{\bigvee_{j=1}^{m_1}(z_{1j} \wedge z'_{1j}):\top}{x'_1}$ is applicable and thus x'_1 is added to E_1. On the other hand, if φ^1 is satisfiable then there exists a model σ of φ^1. Define $\hat{\sigma}$ as the extension of σ defined as $\hat{\sigma}(z'_{1j}) = \neg\sigma(z_{1j})$ for all $1 \leq j \leq m_1$. By virtue of $\sigma \models \varphi^1$ and the construction of $\hat{\sigma}$, we obtain that $\hat{\sigma} \models \psi^1$ while $\hat{\sigma} \not\models \bigvee_{j=1}^{m_1}(z_{1j} \wedge z'_{1j})$. Summarising, φ^1 is unsatisfiable iff $\frac{\bigvee_{j=1}^{m_1}(z_{1j} \wedge z'_{1j}):\top}{x'_1}$ is applicable.

Now suppose that E_i is such that for all $j < i$ the proposition x'_j is contained in E_i iff $\varphi^j[x_1/c_1, \ldots, x_{j-1}/c_{j-1}]$ with $c_1, \ldots, c_{j-1}$ defined as in (1) is unsatisfiable. If $\varphi^i[x_1/c_1, \ldots, x_{i-1}/c_{i-1}]$ is unsatisfiable then any model of the formula

$$\psi^i \wedge \bigwedge_{\substack{1 \leq j < i, \\ c_j = \top}} x_j \wedge \bigwedge_{\substack{1 \leq j < i, \\ c_j = \bot}} x'_j \tag{2}$$

sets both z_{ij} and z'_{ij} for some $1 \leq j \leq m_i$ to $\top$. From (2) and the monotonicity of ψ^i, we obtain that for each model σ' of $\psi^i \wedge \bigwedge_{1 \leq j < i, c_j = \bot} x'_j$ there must exist either an index $1 \leq j < i$ such that σ' sets x_j and x'_j to $\top$, or an index $1 \leq j \leq m_i$ such that σ' sets z_{ij} and z'_{ij} to $\top$. Consequently, $\frac{\bigvee_{j=1}^{m_i}(z_{ij} \wedge z'_{ij}) \vee \bigvee_{j=1}^{i-1}(x_j \wedge x'_j):\top}{x'_i}$ is applicable and $x'_i \in E_{i+1}$. On the other hand, if $\varphi^i[x_1/c_1, \ldots, x_{i-1}/c_{i-1}]$ is satisfiable then there exists a model σ that can be extended to $\hat{\sigma}$ by $\hat{\sigma}(z'_{ij}) = \neg\sigma(z_{ij})$ for all $1 \leq j \leq m_i$ and $\hat{\sigma}(x'_j) = \neg\sigma(x_j)$ for all $1 \leq j < i$ such that $\hat{\sigma} \models \psi^i$ and $\hat{\sigma} \not\models \bigvee_{j=1}^{m_i}(z_{ij} \wedge z'_{ij}) \vee \bigvee_{j=1}^{i-1}(x_j \wedge x'_j)$.

Summarising, φ^i is unsatisfiable iff $\frac{\bigvee_{j=1}^{m_i}(z_{ij} \wedge z'_{ij}) \vee \bigvee_{j=1}^{i-1}(x_j \wedge x'_j):\top}{x'_i}$ is applicable.

The direction from right to left now follows from the fact that φ_n is satisfiable iff $\frac{\bigvee_{j=1}^{m_n}(z_{ij} \wedge z'_{ij}) \vee \bigvee_{j=1}^{n-1}(x_i \wedge x'_i):\top}{\bot}$ is not applicable, which in turn implies that $\langle W, D\rangle$ has a stable extension. Conversely, if $\varphi^n[x_1/c_1, \ldots, x_{n-1}/c_{n-1}]$ is unsatisfiable with $c_1, \ldots, c_{n-1}$ defined as in (1), then any model of $\psi^i \wedge \bigwedge_{1 \leq j < i, \sigma(c_j) = \bot} x'_j$ sets to true either x_j and x'_j for some $1 \leq j < i$ or z_{ij} and z'_{ij} for some $1 \leq j \leq m_i$. As a result, the default $\frac{\bigvee_{j=1}^{m_n}(z_{ij} \wedge z'_{ij}) \vee \bigvee_{j=1}^{n-1}(x_j \wedge x'_j):\top}{\bot}$ is applicable and $\langle W, D\rangle$ does not possess a stable extension. Finally, observe that all formulae contained in $\langle W, D\rangle$ are monotone. Hence, $\langle W, D\rangle$ is a $\{\wedge, \vee, \bot, \top\}$-default theory. To extend the result to $\mathsf{S}_{11} \subseteq [B]$ we proceed as usual and prevent the blowup through transformation into $\vee$-$\wedge$-trees of logarithmic depth.

Thus we have established a reduction from SNSAT to EXT(B) for all B such that $\mathsf{S}_{11} \subseteq [B]$. This concludes the proof. □