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Algorithmic Aspects of Core Logic

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Abstract

This master thesis depicts the advantages of Neil Tennant's Core Logic over other deduction systems considering computational aspects. Being a relevance logic, which usually are undecidable even in the propositional case, one of its features is the **PSPACE**-completeness of its propositional decision problem. This work proves its membership in **PSPACE** by reducing Propositional Core Logic to Propositional Intuitionistic Logic, whereas the given **PSPACE**-hardness proof is based on the model of Statman's **PSPACE**-hardness proof for Propositional Intuitionistic Logic. Therefore, Core Logic is not more complex than non-relevant Intuitionistic Logic. Further, the relevance notion developed by Tennant is strictly proof-theoretical and therefore verifiable by a computer. Accordingly, this paper will answer the question in what way automated proof finding could benefit from this relevance property that provides an effective kind of variable sharing, while still allowing moderate computational complexity of the deduction system.

Zusammenfassung

Die vorliegende Arbeit beschreibt, inwiefern das von Neil Tennant entwickelte Relevanz-Logiksystem Core Logic vorteilhaft für computergestützte Anwendungen ist. Eine Eigenschaft, die Core Logic von anderen Relevanzlogiken abhebt, ist die **PSPACE**-Vollständigkeit seines aussagenlogischen Entscheidungsproblems, deren Beweis das Hauptresultat dieser Arbeit darstellt. Die **PSPACE**-Mitgliedschaft wird gezeigt, indem Propositional Core Logic auf Propositional Intuitionistic Logic reduziert wird, während der **PSPACE**-Schwere-Beweis in Anlehnung an Statmans **PSPACE**-Schwere-Beweis für Propositional Intuitionistic Logic erfolgt. Somit weist Core Logic keine höhere Komplexität auf als nichtrelevante Intuitionistische Logik. Darüber hinaus ist das von Tennant entwickelte Relevance Property rein beweistheoretisch und damit maschinell überprüfbar. Abschließend wird somit die Frage beantwortet, inwiefern computergestütztes logisches Schließen von Tennants Relevanzbegriff profitieren kann, der eine effektive Form des Variable Sharing erfordert und dennoch die moderate Komplexität des Beweissystems zulässt.

Contents

1 Introduction

Motivation

Logic is the mathematical and philosophical science dealing with the structure of valid inference.

A logical system provides a transformation from complex human reasoning to sequences of simple inferences restricted by a finite set of rules. This requires a semantic concept justifying both correctness and soundness of such a system.

Classical Logic is the logical system satisfying the most common understanding of logic. Its underlying semantics require bivalence and extensionality of all objects. It is well known and widely accepted. Many logicians call it *the one right logic*. Still, there are the ones who prefer different systems. They do not agree with certain principles Classical Logic is built on.

Prominent among them are the Intuitionists. They challenge the Principle of Bivalence, the assumption that every sentence must be either *true* or *false*. This leads to rejecting the Law of Excluded Middle. Also the classical semantics of truth valuation cannot be applied. Intuitionists speak of provability instead. A well known semantic concept for this system was developed by Kripke in [Kri63]. It formalizes the approach of growing knowledge by using nodes as states of mind.

Another movement comes from the Relevantists. Their concern is the classical Explosion Principle (from falsehood/contradiction anything follows) that leads to the so called First Lewis Paradox making any sentence ψ derivable from an arbitrary contradiction $\varphi, \neg\varphi$. It is paradoxical because ψ is not necessarily connected to φ. So, a

Relevantist would question the validity of assuming that the absurdity of $\varphi, \neg\varphi$ actually *entails* an arbitrary sentence ψ. Relevance Logics avoid this paradox by ensuring some variable sharing between premises and conclusion; this usually makes their decision problems intractable.

Others, the pluralists, even state that there is no such thing as *the one right logic*. They argue that there are various applications with different requirements that could not all be satisfied by one single logical system.

Core Logic is a logical system developed by Neil Tennant to relevantize Intuitionistic Logic. He stresses that his system preserves relevance from premises to conclusion without increasing the computational complexity compared to Intuitionistic Logic. Another features is that its classicized extension, Classical Core Logic, relevantizes Classical Logic in a corresponding way. It establishes the relevant *core* of logic. That is why Tennant calls it a *core* logic. The present thesis provides a closer look at Tennant's claim.

1.1 Overview

This work starts with a short outline of three established logical systems in Section 1.2. Firstly, Intuitionistic Logic will be described. Its main feature is the rejection of the Law of Excluded Middle resulting in not considering the sentence $\alpha \vee \neg\alpha$ an axiom (Subsection 1.2.1). Secondly, we will have a look at the system of Minimal Logic, which is obtained by rejecting the self-evidence of the intuitionistic (hence also classical) axiom $\neg\alpha \to (\alpha \to \beta)$ (Subsection 1.2.2). Thirdly, the Relevance Logic **R** will be examined, which is one among many Relevance Logics, due to the ambiguity of the concept of relevance. It contains the Law of Excluded Middle but restricts the entailment relation by rejecting Disjunctive Syllogism $((\alpha \vee \beta) \wedge \neg\alpha) \to \beta$ as well as the validity of $\neg\alpha \to (\alpha \to \beta)$ and $\neg\alpha \to (\alpha \to \neg\beta)$ (Subsection 1.2.3).

Subsequently, we will introduce the main characteristics of Core Logic in Chapter 2. In Section 2.1, we will determine Tennant's notion of relevance and what structural preconditions he formulates for Core Logic. In Section 2.2, the reader will find a list of the propositional calculus rules for Core Logic in natural deduction presentation in comparison with the Minimal Calculus rules. Section 2.4 demonstrates how we can inductively extract a proof in the deductive system for Core Logic from a given proof in the Intuitionistic Calculus. Section 2.5 establishes several basic properties of deduction in Core Logic, followed by Chapter 3 in which the proof of **PSPACE**-completeness of Propositional Core Logic will be presented. In Chapter 4, we will discuss the advantages of Core Logic in comparison with other non-classical logics considering automated reasoning before a brief summary and some recommendations for further work will be given in Chapter 5.

1.2 Systems of Non-Classical Logic

Non-classical logics have been developed whenever Classical Logic could not meet the demands of specific territories of reasoning. Some of them extend Classical Logic by adding operators, others restrict it by rejecting certain principles.

The present Section deals with three well known systems of non-classical logic that arose from different camps in the debate over a logical reform.

1.2.1 Intuitionistic Logic

Intuitionism is a philosophical trend concerning the Law of Excluded Middle, which states that every possible sentence either is true or false. Intuitionists gladly accept the Law of Non-Contradiction that states no object can both *be* and *not-be* something at the same time but they endorse the possibility for a proposition not being and not not-being *provable* at the same time. This is where the semantics differ. Intuitionists do not speak of mere *truth*. Their interest aims at *provability* that always requires a *constructive* proof as evidence. There actually are mathematical sentences for which no proof or disproof exists and, according to Gödel's First Incompleteness Theorem (see, e.g., [Raa18]), this is necessarily the case. So, if one evaluates provability of a sentence as being able to present a constructive proof of it, rejecting the Law of Excluded Middle is the indispensable consequence.

The natural deduction system for Intuitionistic Logic **I** contains all rules of Minimal Logic that will be given in Section 2.2 complemented by the absurdity rule Ex Falso Quodlibet (EFQ). This rule in combination with the ¬E rule as stated below represents the equivalence class of contradictory sentences in the sense that if one contradiction is provable, then all contradictory sentences are provable because they are *equivalent*. All non-contradictory sentences φ are obviously provable from $\varphi \wedge \neg\varphi$. This makes every sentence provable from any contradiction.

(¬E) $$\frac{\neg\varphi \quad \begin{matrix}\vdots\\ \varphi\end{matrix}}{\bot}$$ (EFQ) $$\frac{\begin{matrix}\vdots\\ \bot\end{matrix}}{\varphi}$$

To obtain the whole canon of Classical Logic **C**, the Intuitionistic Calculus has to be complemented by one of the four equivalent classical rules of negation ([Ten17, p. 23]).

Dilemma (Dil) $$\dfrac{\begin{matrix}\square\,\overline{\quad}^{\,(i)}\\ \varphi\\ \vdots\\ \psi\end{matrix} \qquad \begin{matrix}\square\,\overline{\quad}^{\,(i)}\\ \neg\varphi\\ \vdots\\ \psi\end{matrix}}{\psi}\,(i)$$

or Classical Reductio (CR) $$\dfrac{\begin{matrix}\Diamond\,\overline{\quad}^{\,(i)}\\ \neg\varphi\\ \vdots\\ \bot\end{matrix}}{\varphi}\,(i)$$

or Double Negation Elimination (DNE) $$\frac{\begin{matrix}\vdots\\ \neg\neg\varphi\end{matrix}}{\varphi}$$

or the Law of Excluded Middle (LEM) $\overline{\varphi \vee \neg\varphi}$.

The boxes next to some of the discharge lines indicate that the discharge of the respective assumption is obligatory. The diamond marks a discharge as permissible, meaning that the rule can be used without discharging any assumptions. This liberty results in Classical Reductio being an application of EFQ: The conclusion can be drawn from any contradiction.

1.2.2 Minimal Logic

Minimal Logic **M** was developed by Ingebrigt Johansson and first published in [Joh37]. His aim was to establish an intuitionistic calculus that would avoid the paradoxes induced by applications of the sentence $\neg\alpha \to (\alpha \to \beta)$ which is an axiom in Intuitionistic Logic as it is in Classical Logic. It can be proved by using the absurdity rule Ex Falso Quodlibet (EFQ).

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{\alpha}^{\,(1)} \qquad \overline{\neg\alpha}^{\,(2)}}{\bot}\,{\scriptstyle \neg E}}{\beta}\,{\scriptstyle EFQ}}{\alpha \to \beta}\,{\scriptstyle \to I(1)}}{\neg\alpha \to (\alpha \to \beta)}\,{\scriptstyle \to I(2)}.$$

Accordingly, EFQ is not available for Minimal Logic.

Minimal Logic furnishes no more than introduction and elimination rules for the connectives and is contained in Intuitionistic Logic. This is what makes it minimal and considerable as a kind of relevant *core* logic. Yet, Minimal Logic allows irrelevance.

Johansson eschews EFQ but the ¬I rule does not require a discharge of assumptions. This makes any negation deducible from absurdity. The proof looks like this:

$$\dfrac{\dfrac{\alpha \quad \neg\alpha}{\bot}\,{\scriptstyle \neg E}}{\neg\beta}\,{\scriptstyle \neg I} \quad ,$$

with the inference of $\neg\beta$ being an application of ¬I. This rule, like EFQ, allows the premises to be irrelevant for the conclusion. This is clearly not wanted in a *relevance* logic. Tennant points out another aspect, that rules out Minimal Logic as a potential *core* logic: Classicizing Minimal Logic—no matter which one of the classical rules of negation one would add—inevitably results in allowing the full canon of Classical Logic including EFQ [Ten17, p. 32 f.]. Core Logic, in contrast, has a classical extension $\mathbb{C}^+$ that classicizes it by adding one of the (relevantized) classical rules of negation and still meets Tennant's relevance condition stated in 2.1.4.

1.2.3 Relevance Logic

Relevance logicians state that for an inference to be valid it is necessary that the premises are somehow relevant to the conclusion. This means that for example the inference *"The moon is made of cheese. Therefore, all lions are cats."* should not be valid because the substance the moon is made of cannot be relevant to the biological classification of mammals. This notion of relevance is of course a semantical one.

The most common proof-theoretical and therefore structural criterion for a logic to be a relevance logic is the variable sharing principle. This means there needs to be a propositional atom that appears in the antecedent as well as in the consequent of an implication. Still, this property is only a necessary condition [Mar14]. There are inferences that satisfy the variable sharing property but still fail to be valid in relevance logics, for example $(\alpha \wedge \neg\alpha) \vdash (\alpha \wedge \beta)$.

By requiring the variable sharing, Relevance Logicians invalidate the classical rule Ex Falso Sequitur Quodlibet (EFQ). This rejection of the equivalence of contradictions is what makes Relevance Logics paraconsistent ("beside the consistent"), meaning that occasional inconsistencies do not entail triviality. Still, it is not necessarily the case that a relevance logic permits true contradictions; this would make it dialethic [Ten05].

Accordingly, there are several systems of Relevance Logic but Tennant particularly references to the Relevance Logic **R** developed by Anderson and Belnap in [AB75] when making his argument for Core Logic as an advance regarding relevance in logic [Ten17, p. 24f].

The system **R** is usually presented in a Hilbert-style axiom system (e.g., [Mar14]).

Axiom	**Name of the Axiom**
$\alpha \to \alpha$	(Identity)
$(\alpha \land \beta) \to \beta$	($\land$E)
$(\alpha \land \beta) \to \alpha$	($\land$E)
$\alpha \to (\alpha \lor \beta)$	($\lor$I)
$\beta \to (\alpha \lor \beta)$	($\lor$I)
$(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$	(Suffixing)
$\alpha \to ((\alpha \to \beta) \to \beta)$	(Assertion)
$(\alpha \to (\alpha \to \beta)) \to (\alpha \to \beta)$	(Contraction)
$((\alpha \to \beta) \land (\alpha \to \gamma)) \to (\alpha \to (\beta \land \gamma))$	($\land$I)
$((\alpha \lor \beta) \to \gamma) \leftrightarrow ((\alpha \to \gamma) \land (\beta \to \gamma))$	($\lor$E)
$(\alpha \land (\beta \lor \gamma)) \to ((\alpha \land \beta) \lor (\alpha \land \gamma))$	(Distribution)
$(\alpha \to \neg\beta) \to (\beta \to \neg\alpha)$	(Contraposition)
$\neg\neg\alpha \to \alpha$	(Double Negation Elimination)

Additionally, only two rules of inference are necessary.

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi}\,\text{(Adjunction)} \qquad \text{and} \qquad \frac{\varphi \quad \varphi \rightarrow \psi}{\psi}\,\text{(Detachment)} \;.$$

For the reason that all other logical systems in this work are presented by giving a list of natural deduction rules, we will give such a representation of **R** below, which was taken from [Ten05].

If we look at $\vee$E above or at the elimination rule for $\vee$ in the rule presentation below, we can see how Disjunctive Syllogism fails in **R**. Without using the absurdity rule EFQ, there is no option to infer an overall conclusion from only one disjunct if the other one leads to absurdity. This is a genuine disadvantage of this system. In [Ten17, p. 115], Tennant stresses the value of Disjunctive Syllogism for mathematics. In his point of view, a *core* logic could not do without it. This is why he relaxed the $\vee$E rule in his system to allow the logician to turn towards the other disjunct once the first led to absurdity.

	Introduction	Elimination
$\neg$	$\dfrac{\begin{array}{c}\Box\,\overline{\varphi}\,(i) \\ \vdots \\ \bot\end{array}}{\neg\varphi}\,(i)$	$\dfrac{\neg\varphi \quad \begin{array}{c}\vdots \\ \varphi\end{array}}{\bot}$
$\wedge$	$\dfrac{\begin{array}{c}\Delta \\ \vdots \\ \varphi\end{array} \quad \begin{array}{c}\Delta \\ \vdots \\ \psi\end{array}}{\varphi \wedge \psi}$	$\dfrac{\varphi \wedge \psi \qquad \begin{array}{c}(i)\,\overline{\varphi}\,,\ \Box\,\overline{\psi}\,(i) \\ \vdots \\ \theta\end{array}}{\theta}$
$\vee$	$\dfrac{\begin{array}{c}\vdots \\ \varphi\end{array}}{\varphi \vee \psi}$ $\qquad$ $\dfrac{\begin{array}{c}\vdots \\ \psi\end{array}}{\varphi \vee \psi}$	$\dfrac{\varphi \vee \psi \qquad \begin{array}{c}\Delta, \Box\,\overline{\varphi}\,(i) \\ \vdots \\ \theta\end{array} \qquad \begin{array}{c}\Delta, \Box\,\overline{\psi}\,(i) \\ \vdots \\ \theta\end{array}}{\theta}\,(i)$
$\to$	$\dfrac{\begin{array}{c}\Box\,\overline{\varphi}\,(i) \\ \vdots \\ \psi\end{array}}{\varphi \to \psi}\,(i)$	$\dfrac{\varphi \to \psi \quad \begin{array}{c}\vdots \\ \varphi\end{array} \quad \begin{array}{c}\Box\,\overline{\psi}\,(i) \\ \vdots \\ \theta\end{array}}{\theta}\,(i)$

Distributivity

$$\frac{\varphi \wedge (\psi \vee \theta)}{(\varphi \wedge \psi) \vee (\varphi \wedge \theta)}$$

Classical Reductio

$$\dfrac{\begin{array}{c} \Box\ \overline{\neg\varphi}^{\,(i)} \\ \vdots \\ \bot \end{array}}{\varphi}\,(i)$$

2 Core Logic

Core Logic ($\mathbb{C}$) is a logical system developed by Neil Tennant over the previous four decades. The first publication dealing with his notion of relevance and the transitivity of the deducibility relation was [Ten78]. Subsequently, Tennant treated the suitability of his Intuitionistic Relevance Logic **IR** for automated reasoning in *Autologic* [Ten92]. In a proceeding work, the system was renamed *Core Logic* $\mathbb{C}$ [Ten12].

One main feature of Core Logic is that it relevantizes Classical Logic in the same way as it does Intuitionistic Logic. This is why Tennant named it *Core Logic*; because it fits in the center of the most common logical systems being their relevant *core*.

> *Core Logic is the minimal inviolable core of logic without any part of which one would not be able to establish the rationality of belief-revision.* ([Ten17, p. 48])

Relevantizing in Tennant's notion is achieved by adjusting certain logical rules by generalizing them and getting rid of the absurdity rule. Furthermore, in the natural deduction setting all major premises for elimination (MPE) must *stand proud*, meaning they cannot have any proof work above them. In the sequent calculus, which is available for Core Logic as well, one has to get rid of the structural rules *Thinning* and *Cut*. These limitations also lead to a welcome isomorphism between sequent proofs and natural deductions, which is not going to be elucidated further in the present paper.

Tennant's maxim is to modify the definition of the *deducibility* relation rather than redefining *entailment* as other relevance logicians suggest. This leads to a restriction of the Deduction Theorem as will be explained in Subsection 2.1.1.

2.1 Relevance in Core Logic

Tennant stresses the importance of eschewing the so called First Lewis Paradoxes (i.e., $A, \neg A \vdash B$ and $A, \neg A \vdash \neg B$) to preserve relevance in reasoning [Ten05].

Especially in automated theorem proving it is crucial to avert curious inferences produced by applications of EFQ. It is a known problem with databases that inconsistencies appear caused by mistakes or multiple sourcing. Tennant suggests to value the inference of inconsistency as much as the inference of the sought-after conclusion.

The deducibility relation can be denoted as a sequent as developed by Gentzen in [Gen35]. In the present work we will write the premises as comma separated conjuncts Δ followed by a colon and the single sentence conclusion φ.

The Compactness Theorem in its usual interpretation allows us to subset down on the left side of the colon. This means that, if the question is whether a conclusion follows from a (possibly infinite) set of premises, then we search for a derivation of the conclusion from some finite subset of the set of premises. Tennant argues that, in fact, it should be possible to subset down on both sides, that is allowing the conclusion to be φ or $\perp$, where $\perp$ is to be read as an abbreviation for $\emptyset$. If there is nothing else to be had from a set of premises, then we should take the knowledge of its inconsistency as epistemic gain [Ten05, p. 11].

Definition 2.1. $\Delta : \Gamma$ is a subsequent of $\Delta' : \Gamma'$ (abbreviated $\Delta : \Gamma \sqsubseteq \Delta' : \Gamma'$) if and only if $\Delta \subseteq \Delta'$ and $\Gamma \subseteq \Gamma'$.

This assumption leads to redefining the decision problem for an input sequent $\Delta : \varphi$, as being positively settled by a proof of any subsequent of $\Delta : \varphi$, explicitly including $\Gamma : \perp$ with $\Gamma \subseteq \Delta$ [Ten92, p. 189].

2.1.1 The Deduction Theorem

Tennant's main cause of developing a new logical system can be recapitulated by the following quote:

> *The case is straightforward. We wish to avoid the positive and negative forms of the First Lewis Paradox: 'A, ¬A, so B'; and 'A, ¬A so ¬B'. Nothing could be simpler by way of motivation. Avoid those, and everything falls into place[...].* ([Ten17, p.13])

And yet he does not aim at invalidating $\neg\alpha \to (\alpha \to \beta)$ and $\neg\alpha \to (\alpha \to \neg\beta)$, as Anderson and Belnap did, but at reflecting upon the deducibility relation independently from the object-linguistic conditional.

The Deduction Theorem as stated in Classical Logic is as follows:

$$\Delta, \varphi \vdash_C \psi \Leftrightarrow \Delta \vdash_C \varphi \to \psi.$$

Tennant accepts the material conditional as a connective and does not challenge its truth table. It is valid in Core Logic to infer $\varphi \to \psi$ from the contradictoriness of φ. Of course, there is a connection between premises and conclusion via φ. But at the point that φ is detached from the conclusion, the inference forfeits relevance. For example, $\neg\varphi \vdash_{\mathbb{C}} \varphi \to \psi$ is a valid inference, yet $\neg\varphi, \varphi \nvdash_{\mathbb{C}} \psi$. Consequently, to obtain what Tennant calls relevance *"at the level of the turnstile"*(see, e.g., [Ten17, p. 15]), the "$\Leftarrow$"-direction of the Deduction Theorem is not valid in Core Logic. If, on the other hand, φ is not contradictory, we can be assured that either φ or some other member of the set of premises is relevant to ψ and adding φ to the set of premises would not violate the requirement of relevance. So, from this meta-perspective, the reverse Deduction Theorem can be regarded admissible for non-contradictory antecedents only [Ten17, p. 46].

2.1.2 Disjunctive Syllogism

As mentioned in 1.2.3, Anderson and Belnap banned Disjunctive Syllogism in their Relevance Logic **R**. This, according to Tennant, makes **R** not appropriate for mathematical reasoning. It is crucial for mathematicians to infer whatever one disjunct entails if the other disjunct leads to absurdity. Yet Disjunctive Syllogism is one of the two properties of logical systems that allow the First Lewis Paradox to be provable. Tennant states this proof as follows:

> *Suppose A. Then by (∨I) we have $A \vee B$. Now suppose ¬A. By Disjunctive Syllogism, we have B. Hence, by Unrestricted Transitivity of Deduction, we have B following from A, ¬A.* ([Ten17, p. 264])

Or formal:

$$\frac{A : A \vee B \quad \neg A, A \vee B : B}{A, \neg A : B}\text{Cut}$$

So, the situation is as follows: Disjunctive Syllogism and Unrestricted Transitivity of Deducibility lead to unwanted results; therefore, we must give up Disjunctive Syllogism or Unrestricted Transitivity of Deducibility. Now, Tennant argues, it would be quite ironic to give up Disjunctive Syllogism, employing this same principle in ones argument. So, we cannot give up Disjunctive Syllogism; hence, we must give up Unrestricted Transitivity of Deducibility. [Ten17, p. 265]

2.1.3 Thinning and Cut

As follows from the argument made in Subsection 2.1.2, Core Logic does not contain Cut as a rule within the system. Still, Tennant formulates a restricted epistemically gainful version of Cut illustrating the restricted transitivity of Core-deducibility.

$$\text{Cut}\,\frac{\Gamma \vdash_C \varphi \quad \varphi, \Delta \vdash_C \psi}{\Gamma, \Delta \vdash_C \psi} \qquad \text{Restricted Cut}\,\frac{\Gamma \vdash_{\mathbb{C}} \varphi \quad \varphi, \Delta \vdash_{\mathbb{C}} \psi}{\Sigma \vdash_{\mathbb{C}} \psi'},$$

with Γ and Δ sets of premises, φ and ψ single sentences and $\Sigma \subseteq \Gamma \cup \Delta$ and $\psi' \subseteq \psi$.

So, the lack of the rule of Cut in $\mathbb{C}$ is a consequence of the restriction of transitivity. Yet it is possible to combine two proofs by a normalization process that restructures the combined proof to meet the requirements of the Core normal form (see Definition 2.5).

If the union of the premise sets Γ and Δ is inconsistent, this will be revealed; the conclusion of the normalized resulting proof will be $\bot$ (see [Ten12]).

The other structural rule that $\mathbb{C}$ does not furnish is the Thinning Rule. In the Classical Sequent Calculus, there is Thinning on the Right (TR) and Thinning on the Left (TL).

$$\frac{\Gamma \vdash_{C} \varphi}{\Gamma \vdash_{C} \varphi, \psi}\,(\text{TR}) \qquad\qquad \frac{\Gamma \vdash_{C} \varphi}{\Gamma, \psi \vdash_{C} \varphi}\,(\text{TL})$$

Tennant allows in his sequent calculus for Core Logic the conclusion to be a singleton at most which rules out Thinning on the Right. Thinning on the Left jeopardizes the required guaranteed consistency of the set of premises. Hence, thinnings are not admissible for Core Logic [Ten17, p. 45].

2.1.4 The Relevance Condition

The relevance condition introduced by Tennant in [Ten92] and reviewed in [Ten17] is a variable sharing property that holds for Core Logic as well as for Classical Core Logic.

The common variable sharing principle requires only one propositional variable to occur on both sides of the entailment relation (e.g., [Rea88, p. 121]). Tennant, on the other hand, constructs a concept of relevance that requires *all* premises to be relevant to the conclusion. This is achieved by defining chains of connected sentences called $\bowtie$-*components*. These are graphs of premises connected among themselves with two sentences being connected if and only if they share a variable with opposite signs. The relevance property requires at least one of the sentences from each of these $\bowtie$-components to share a propositional variable of equal sign with the conclusion. Below, we will give the formal definition of this relevance property (see [Ten17, p. 266 ff.]).

Definition 2.2. An atom is a propositional variable.

Remark: $\bot$ is not an atom. $\bot$ never occurs as a subformula; it is merely a punctuation device in proofs, used in order to register absurdity.

Definition 2.3. We use the expression $\chi \preceq^{+} \theta$ to mean that the subsentence occurrence χ is a positive one within the sentence θ. Likewise, $\chi \preceq^{-} \theta$ means that the subsentence occurrence χ is a negative one within the sentence θ. The co-inductive definition of these relations is as follows.

(1) Every sentence is a positive subsentence occurrence in itself

(2) All positive [resp., negative] subsentence occurrences in φ are negative [resp., positive] subsentence occurrences in $\neg\varphi$.

(3) All positive [resp., negative] subsentence occurrences in φ and ψ are positive [resp., negative] subsentence occurrences in $(\varphi \wedge \psi)$.

(4) All positive [resp., negative] subsentence occurrences in φ and ψ are positive [resp., negative] subsentence occurrences in $(\varphi \vee \psi)$.

(5) All positive [resp., negative] subsentence occurrences in ψ are positive [resp., negative] subsentence occurrences in $(\varphi \rightarrow \psi)$.

(6) All positive [resp., negative] subsentence occurrences in φ are negative [resp., positive] subsentence occurrences in $(\varphi \rightarrow \psi)$.

(7) If $\chi \preceq^{+} \theta$ then this can be shown by clauses (1)–(6)

(8) If $\chi \preceq^{-} \theta$ then this can be shown by clauses (1)–(6)

$\varphi \underset{A}{\approx} \psi \equiv_{df}$ the atom A has occurrences of the same parity in φ and in ψ; that is, either $(A \preceq^{+} \varphi$ and $A \preceq^{+} \psi)$ or $(A \preceq^{-} \varphi$ and $A \preceq^{-} \psi)$.

$\varphi \approx \psi \equiv_{df}$ for some atom A we have $\varphi \underset{A}{\approx} \psi$.

$\varphi \approx \Delta \equiv_{df}$ for some ψ in Δ, we have $\varphi \approx \psi$; that is, some atom has the same parity (positive or negative, at some occurrence) in φ as it has at some occurrence in some member of Δ.

$\varphi \underset{A}{\bowtie} \psi \equiv_{df}$ the atom A has occurrences of opposite parities in φ and in ψ; that is, either $(A \preceq^{+} \varphi$ and $A \preceq^{-} \psi)$ or $(A \preceq^{-} \varphi$ and $A \preceq^{+} \psi)$.

$\varphi \bowtie \psi \equiv_{df}$ for some atom A we have $\varphi \underset{A}{\bowtie} \psi$.

$\pm\varphi \equiv_{df} \varphi \bowtie \psi$.

A sequence $\varphi_1, ..., \varphi_n$ ($n > 1$) of pairwise distinct sentences is a $\bowtie$-path connecting φ_1 to φ_n in $\Delta \equiv_{df}$ for $1 \leq i \leq n$, φ_i is in Δ, and for $1 \leq i < n$, $\varphi_i \bowtie \varphi_{i+1}$.

φ and ψ are $\bowtie$-connected in Δ (in symbols: $\varphi \underline{\bowtie} \psi$) $\equiv_{df}$ if $\varphi \neq \psi$, then there is a $\bowtie$-path connecting φ to ψ in Δ.

A set Δ of formulas is $\bowtie$-connected $\equiv_{df}$ for all φ, ψ in Δ, if $\varphi \neq \psi$, then $\varphi \underline{\bowtie} \psi$.

A $\bowtie$-component of Δ is a non-empty, inclusion-maximal $\bowtie$-connected subset of Δ (where the $\bowtie$-connections are established via members of Δ).

$\varphi \blacktriangleleft \Delta \equiv_{df}$ for every $\bowtie$-component Γ of Δ, $\varphi \approx \Gamma$.

Remark: We cannot have $\perp \blacktriangleleft \Delta$ since $\perp$ is not a sentence.

Suppose $\Delta \neq \emptyset$. Then

$$\sharp\Delta \equiv_{df} \begin{cases} \text{if } \Delta \text{ is a singleton, say } \{\delta\}, \text{ then } \pm\delta; \\ \text{and} \\ \text{if } \Delta \text{ is not a singleton, then } \Delta \text{ is } \bowtie\text{-connected.} \end{cases}.$$

We shall say that a set Δ of premises is relevantly connected both within itself and to a conclusion φ (in symbols: $\mathcal{R}(\Delta, \varphi)$) just in case exactly one of the following three conditions is satisfied:

1. Δ is non-empty, φ is $\perp$, and $\sharp\Delta$.
2. Δ is non-empty, φ is not $\perp$, and $\varphi \blacktriangleleft \Delta$.
3. Δ is empty, φ is not $\perp$, and $\pm\varphi$.

To sum up the aforementioned criteria for Tennant's relevance property, we say that a deduction $\Delta \vdash \varphi$ is relevant if only if one of the following cases applies:

1. If φ is $\perp$, either all sentences in Δ are $\bowtie$-connected or Δ is a singleton containing a positive and a negative occurrence of some atom A.

2. If Δ is not empty and φ is not $\perp$, all $\bowtie$-components Γ of Δ contain an atom A that has the same parity in Γ as it has in φ.
3. If Δ is empty, φ is a singleton containing a positive and a negative occurrence of some atom A.

Tennant claims his relevance property to be the *"most exigent such property formulated thus far"* [Ten17, p. 279]. Because of the inclusion of **R** in $\mathbb{C}^+$, it also holds for **R** and its subsystems.

A noteworthy fact is that Tennant limits the necessity for relevance to the deducibility relation, allowing the truth table of the conditional to stay unaffected. This will be apparent in the following definition of the Core rules in Section 2.2.

2.2 Core Rules

The below list gives an overview of the deduction rules of Core Logic in comparison to the well known rules of the Minimal Calculus of Minimal Logic. As the reader will notice, Tennant makes very little changes to obtain the Core rules from the Minimal rules. It is worth mentioning that these changes do not compromise the established meanings of the logical operators.

Definition 2.4. An elimination of the dominant connective of a sentence is in *parallelized* form if and only if the elimination of the dominant connectives of its subsentences are made in its subproofs.

This means that all elimination work that could be applied on subformulas of the Major Premise for Elimination (MPE) is done in the elimination's subproofs. The proof is built *bottom-up*, starting with a formula φ, and—instead of just eliminating the dominant connective to infer a subformula ψ—the respective subformula ψ is made the assumption of a subproof. The elimination with MPE φ discharges ψ and concludes with the conclusion of the elimination with MPE ψ. An example is given below contrasting the parallelized form with the common serial form.

Parallel

$$\dfrac{(A \to C) \land B \qquad \dfrac{\overline{A \to C}^{\,(1)} \quad \begin{matrix} \vdots \\ A \end{matrix} \quad \begin{matrix} \overline{C}^{\,(2)} \\ \vdots \\ D \end{matrix}}{D}\,{\to}\mathrm{E}(2)}{D}\,{\land}\mathrm{E}(1)$$

Serial

$$\begin{matrix} \dfrac{\dfrac{(A \to C) \land B}{A \to C}\,{\land}\mathrm{E} \quad \begin{matrix} \vdots \\ A \end{matrix}}{C}\,{\to}\mathrm{E} \\ \vdots \\ D \end{matrix}$$

Apparently, in the left proof tree the inner elimination (→E) is placed in a subproof above the outer elimination (∧E). Furthermore, the →E contains a subproof of D from C representing additional proof steps applied on C concluding with D. This structure forces the elimination of the dominant connective to be the bottom-most elimination step. In the serial form, on the other hand, the elimination of the dominant connective is the top-most elimination step as is depicted in the right proof tree.

Definition 2.5. We say a proof is in *Core normal form* if it meets the following requirements.

1. All major premises for elimination (MPE) stand proud, i.e., they are not the conclusion of preceding proof work.
2. All eliminations are in parallelized form.

Note that the discharge rules distinguish between *obligatory* and *permitted* discharges. A box in front of an inference line marks an obligatory discharge meaning that the assumption actually has to be used for the inference. A diamond in contrast labels a permitted (also referred to as *vacuous*) discharge, allowing the rule to be applied even if the premise has not been used to reach the conclusion.

The admissibility of vacuous discharge makes Minimal (hence Intuitionistic and Classical) ¬I actually an application of EFQ. The same applies to Intuitionistic and Classical →I in the case of a contradictory antecedent. This is why Tennant restricts those rules as to be only applicable if the assumption actually has been used. Regarding the ∧E rule, the box means that at least one of the conjuncts must have been made an assumption of the subproof.

The conclusion $\theta/\bot$ of the Core rule $\vee$E is to be read as *either* θ *or* $\bot$ while the overall $\vee$E concludes with $\bot$ if and only if all of its subproofs end with $\bot$.

Minimal Logic | Core Logic

(¬ I)

$$\dfrac{\begin{array}{c}\diamond\,\overline{\ \varphi\ }^{\,(i)}\\ \vdots\\ \bot\end{array}}{\neg\varphi}\,(i) \qquad\qquad \dfrac{\begin{array}{c}\square\,\overline{\ \varphi\ }^{\,(i)}\\ \vdots\\ \bot\end{array}}{\neg\varphi}\,(i)$$

(¬ E)

$$\dfrac{\neg\varphi \quad \begin{array}{c}\vdots\\ \varphi\end{array}}{\bot} \qquad\qquad \dfrac{\neg\varphi \quad \begin{array}{c}\vdots\\ \varphi\end{array}}{\bot}$$

(∧ I)

$$\dfrac{\begin{array}{c}\vdots\\ \varphi\end{array} \quad \begin{array}{c}\vdots\\ \psi\end{array}}{\varphi\wedge\psi} \qquad\qquad \dfrac{\begin{array}{c}\vdots\\ \varphi\end{array} \quad \begin{array}{c}\vdots\\ \psi\end{array}}{\varphi\wedge\psi}$$

(∧ E)

$$\dfrac{\varphi\wedge\psi \qquad \begin{array}{c}{}^{(i)}\overline{\ \varphi\ }\,,\ \square\,\overline{\ \psi\ }^{\,(i)}\\ \vdots\\ \theta\end{array}}{\theta} \qquad\qquad \dfrac{\varphi\wedge\psi \qquad \begin{array}{c}{}^{(i)}\overline{\ \varphi\ }\,,\ \square\,\overline{\ \psi\ }^{\,(i)}\\ \vdots\\ \theta\end{array}}{\theta}$$

Minimal Logic | Core Logic

(∨ I)

Minimal Logic:

$$\dfrac{\begin{array}{c}\vdots\\ \varphi\end{array}}{\varphi \vee \psi} \qquad \dfrac{\begin{array}{c}\vdots\\ \psi\end{array}}{\varphi \vee \psi}$$

Core Logic:

$$\dfrac{\begin{array}{c}\vdots\\ \varphi\end{array}}{\varphi \vee \psi} \qquad \dfrac{\begin{array}{c}\vdots\\ \psi\end{array}}{\varphi \vee \psi}$$

(∨ E)

Minimal Logic:

$$\dfrac{\varphi \vee \psi \quad \begin{array}{c}\Box\ \overline{\ \varphi\ }^{\,(i)}\\ \vdots\\ \theta\end{array} \quad \begin{array}{c}\Box\ \overline{\ \psi\ }^{\,(i)}\\ \vdots\\ \theta\end{array}}{\theta}\,(i)$$

Core Logic:

$$\dfrac{\varphi \vee \psi \quad \begin{array}{c}\Box\ \overline{\ \varphi\ }^{\,(i)}\\ \vdots\\ \theta/\bot\end{array} \quad \begin{array}{c}\Box\ \overline{\ \psi\ }^{\,(i)}\\ \vdots\\ \theta/\bot\end{array}}{\theta/\bot}\,(i)$$

(→ I)

Minimal Logic:

$$\dfrac{\begin{array}{c}\Diamond\ \overline{\ \varphi\ }^{\,(i)}\\ \vdots\\ \psi\end{array}}{\varphi \to \psi}\,(i)$$

Core Logic:

$$\dfrac{\begin{array}{c}\Diamond\ \overline{\ \varphi\ }^{\,(i)}\\ \vdots\\ \psi\end{array}}{\varphi \to \psi}\,(i) \qquad \dfrac{\begin{array}{c}\Box\ \overline{\ \varphi\ }^{\,(i)}\\ \vdots\\ \bot\end{array}}{\varphi \to \psi}\,(i)$$

(→ E)

Minimal Logic:

$$\dfrac{\varphi \to \psi \quad \begin{array}{c}\vdots\\ \varphi\end{array} \quad \begin{array}{c}\Box\ \overline{\ \psi\ }^{\,(i)}\\ \vdots\\ \theta\end{array}}{\theta}\,(i)$$

Core Logic:

$$\dfrac{\varphi \to \psi \quad \begin{array}{c}\vdots\\ \varphi\end{array} \quad \begin{array}{c}\Box\ \overline{\ \psi\ }^{\,(i)}\\ \vdots\\ \theta\end{array}}{\theta}\,(i)$$

2.3 Inclusion Between Logical Systems

The aforementioned systems can be separated by the theorems that they prove. Figure 2.1 gives an overview of the inclusions among those systems.

It is worth mentioning that the only theorem that separates Core Logic (resp. Classical Core Logic) from Intuitionistic Logic (resp. Classical Logic), the system it relevantizes, is the First Lewis Paradox.

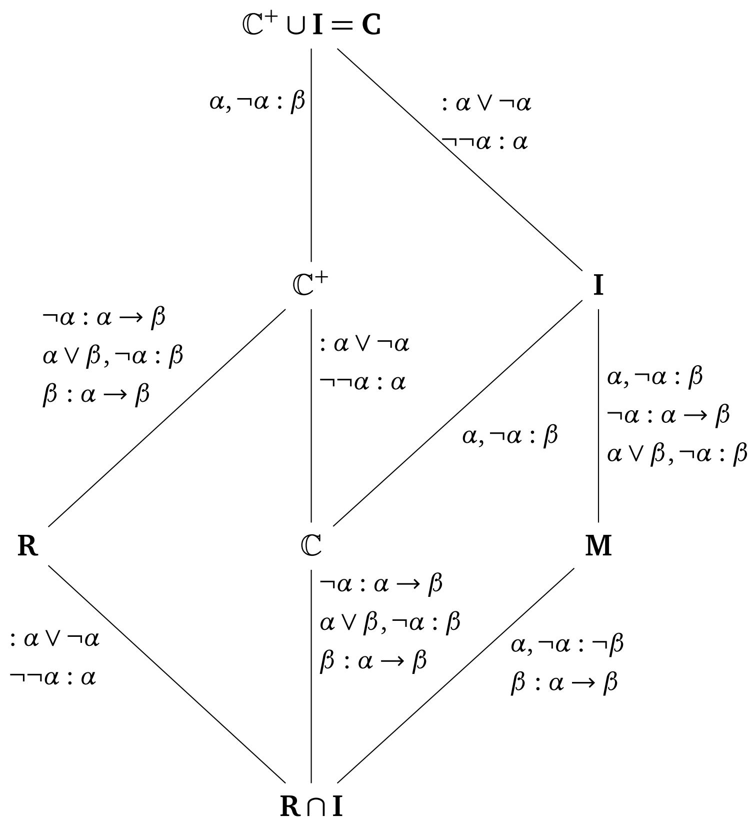

Figure 2.1: Inclusions of logical systems described in this thesis

2.4 Obtaining Core Proofs from Intuitionistic Proofs

An important quality of Core Logic is what Tennant calls the Extraction Theorem (2.6). It states that a Core proof can be *extracted* from an Intuitionistic proof.

Remark: Core Logic is based on a different Completeness Theorem. The common requirement for a proof system S to be complete is as follows:

$\Delta \models \varphi \Rightarrow \Delta' \vdash_S \varphi$ with finite $\Delta' \subseteq \Delta$.

Tennant's notion of completeness is the following:

$\Delta \models \varphi \Rightarrow \Delta' \vdash_S \varphi'$ with finite $\Delta' \subseteq \Delta$ and $\varphi' \subseteq \varphi$. Where the conclusion φ is a singleton and φ' is either φ or $\perp$ ([Ten17, p. 6 f.]).

Theorem 2.6. *There is an algorithm E such that for any (normal) intuitionistic proof Π of $\Delta : \psi$, E with input Π returns a Core proof of some subsequent of $\Delta : \psi$.*

This means that any Intuitionistic proof of ψ from a set of premises Δ can be converted into a Core proof of ψ or of $\perp$ from some subset of Δ.

To obtain a Core proof from an Intuitionistic proof we assume the **I**-proof to be in Core normal form (see Definition 2.5). This is the form in which Tennant introduces not only the Core rules, but also the rules of the Minimal Calculus in [Ten92] and [Ten17].

> *There is another important feature of our formulation of the rules for **M** and for **IR** [ℂ]. Major premisses for elimination 'stand proud' in our proof trees; they never stand as conclusions of any rules.* ([Ten92, p. 41])

Now we extract the Core proof from the intuitionistic proof by scanning it for applications of the absurdity rule and find relevant deductions of the wanted conclusion or of $\perp$ from a subset of the used set of premises.

Proof. By induction on the proof complexity of the intuitionistic proof Π.

Base case: $\alpha : \alpha$ is trivial.

Induction hypothesis: The Extraction Theorem holds for all proofs less complex than Π.

Induction step: The only rules that need to be examined are ∨E, ¬I and →I. All other rules are the same in Core Logic as in Intuitionistic Logic. Applications of EFQ are cut off and marked with ⊥. So, we have three cases to be considered depending on the rule last applied in Π.

1. ∨E: The intuitionistic proof is of the following form.

$$\dfrac{\alpha \vee \beta \qquad \begin{array}{c} \Box\,\overline{\alpha}^{\,(i)} \\ \vdots \\ \gamma \end{array} \qquad \begin{array}{c} \Box\,\overline{\beta}^{\,(i)} \\ \vdots \\ \gamma \end{array}}{\gamma}\,(i)$$

   By induction hypothesis, the subproofs of γ from each disjunct can be converted to Core proofs of γ or ⊥. So, the overall conclusion by Core rules is γ if at least one of the Core subproofs leads to γ, or is ⊥ if both Core subproofs lead to ⊥.

2. ¬I: If the conclusion is $\neg\alpha$ and α has actually been discharged in the **I**-proof we can infer $\neg\alpha$ in Core. Otherwise if absurdity was derived from the set of premises with no use of α we infer ⊥.

3. →I: The intuitionistic proof tree is of the following form.

$$\dfrac{\begin{array}{c} \Delta, \ \Diamond\,\overline{\alpha}^{\,(i)} \\ \vdots \\ \beta \end{array}}{\alpha \to \beta}\,(i)$$

   By induction hypothesis the subproof of β from $\Delta \cup \{\alpha\}$ can be converted to a corresponding Core proof of either β or ⊥ from some subset Γ of $\Delta \cup \{\alpha\}$.

   a) If the conclusion of the Core subproof is β, by Core rules we infer $\alpha \to \beta$.

b) If the conclusion of the Core subproof is $\bot$, by Core rules we infer $\alpha \to \beta$ if α is in Γ (i.e., α has been used to get to the contradiction), or, if the discharge in the **I**-proof was vacuous, we infer $\bot$.

□

The extraction process can be done in a straightforward way by searching for applications of the absurdity rule, including applications of ¬I and →I with vacuous discharge. The latter two involve checking if the discharge requirements for Core Logic are met, which takes at most a search through all undischarged assumptions. This cannot exceed the length of the intuitionistic proof Π. Checking an application of ∨-elimination takes at most a search through its minor premises. No more than one of these searches has to be done for each step of Π. So the complexity of extraction is quadratic in the length of Π. [Ten92, p. 189 f.]

Algorithm 2.1: Extract Core proof from intuitionistic proof

Data: An intuitionistic proof in natural deduction tree representation
Result: The corresponding Core proof

```
post order search the whole tree
    if curNode == ⊥ then
        if curRule == ¬I then
            // positive:  (¬ x) ↦ x
            if NOT (positive(curNode.parent) is discharged in this step) then
                curNode.parent := ⊥;
        if curRule == →I then
            // antecedent:  (x → y) ↦ x
            if NOT (antecedent(curNode.parent) is discharged in this step) then
                curNode.parent := ⊥;
        if curRule == ∨E then
            /* check if at least one of the minor premises leads
               to the conclusion; the major premise cannot be ⊥
               because it has no proof work above it             */
            if At least two siblings of curNode != ⊥ then
                continue;
            curNode.parent := ⊥;
        if curNode.parent == NIL then
            // the root
            continue;
        else
            // EFQ
            curNode.parent := ⊥;
end;
```

2.5 Basic Proof-Theoretic Properties of Core Logic

Lemma 2.7. *If a propositional sentence φ is classically provable from a set of premises Γ, then either $\neg\neg\varphi$ or $\bot$ is Core deducible from Δ with $\Delta \subseteq \Gamma$.*

Proof. We know by Glivenko's Theorem that for propositional logic $\Gamma \vdash_C \varphi \Leftrightarrow \Gamma \vdash_I \neg\neg\varphi$. Now Lemma 2.7 is obvious by the Extraction Theorem 2.6. □

Corollary 2.8. *If Γ is a consistent set (i.e., $\bot$ is not deducible from Γ), then $\neg\neg\varphi$ is deducible from Γ in Core Logic if and only if φ is deducible from Γ in Classical Logic.*

Lemma 2.9. *If $(\varphi \to \psi)$ is Core deducible from Γ, then either ψ or $\bot$ is Core deducible from $\Gamma \cup \{\varphi\}$.*

Proof. Trivial by cases of the Core rule $\to$I. □

Corollary 2.10. *If $\bot$ cannot be deduced from $\Gamma \cup \{\varphi\}$ (i.e., $\Gamma \cup \{\varphi\}$ is consistent) and $(\varphi \to \psi)$ is Core deducible from Γ, then ψ is Core deducible from $\Gamma \cup \{\varphi\}$.*

Lemma 2.11. *If θ is Core deducible from $\Gamma \cup \{(\varphi \vee \psi)\}$ and neither φ nor ψ are contradictory, then θ is Core deducible from $\Gamma \cup \{\varphi\}$ and also from $\Gamma \cup \{\psi\}$.*

Proof. In Core Logic $\vee$-elimination allows one disjunct to entail absurdity to preserve Disjunctive Syllogism without using EFQ. So we have three possible cases on the right side of the following implication:

$\Gamma \cup \{(\varphi \vee \psi)\} \vdash_{\mathbb{C}} C \Rightarrow$

1. $\Gamma \cup \{\varphi\} \vdash_{\mathbb{C}} \theta$and $\Gamma \cup \{\psi\} \vdash_{\mathbb{C}} \theta$ or
2. $\Gamma \cup \{\varphi\} \vdash_{\mathbb{C}} \theta$ and $\Gamma \cup \{\psi\} \vdash_{\mathbb{C}} \bot$ or
3. $\Gamma \cup \{\varphi\} \vdash_{\mathbb{C}} \bot$ and $\Gamma \cup \{\psi\} \vdash_{\mathbb{C}} C$.

So, if $\bot$ cannot be deduced from either $\Gamma \cup \{\varphi\}$ or from $\Gamma \cup \{\psi\}$, the first possibility is necessarily the case. □

Lemma 2.12. *If θ is Core deducible from $\Gamma \cup \{\varphi\}$ and from $\Gamma \cup \{\psi\}$, it is Core deducible from $\Gamma \cup \{(\varphi \vee \psi)\}$.*

Proof. Trivial by inspection of the $\vee$E rule of Core Logic. □

Lemma 2.13. *If* Γ *contains no formula containing* $\vee$*, then*

$$\Gamma \vdash_{\mathbb{C}} \varphi \vee \psi \Rightarrow \Gamma \vdash_{\mathbb{C}} \varphi \text{ or } \Gamma \vdash_{\mathbb{C}} \psi. \tag{2.1}$$

Proof. By induction on the length of proof.

Base cases: $\varphi \vdash_{\mathbb{C}} \varphi \vee \psi$ and $\psi \vdash_{\mathbb{C}} \varphi \vee \psi$. To infer φ or ψ all we have to do is cut off the proof.

Induction Hypothesis: The Lemma 2.13 holds for all proofs less complex than the one considered.

Induction Step: The rule applied can only be $\vee$I or one of the elimination rules $\vee$E, $\wedge$E or $\rightarrow$E.

1. $\vee$I: φ or ψ need to be available as a premise. So to infer φ or ψ we need to cut off the $\vee$I.
2. $\vee$E: In Core Logic all major premises for elimination stand proud and therefore this particular MPE of our $\vee$E would have to be in Γ. But Γ does not contain any occurrence of $\vee$. Contradiction.
3. $\wedge$E: If the disjunction is the conclusion of an application of $\wedge$E it has to be inferred directly from one of the conjuncts. This means that one of the conjuncts either has $\varphi \vee \psi$ as a subformula, which is assumed not to be the case, or leads to it using $\vee$I in one of its subproofs on either φ or ψ. So, φ or ψ can be inferred by substituting all occurrences of $\varphi \vee \psi$ with the respective.
4. $\rightarrow$E: If the disjunction is the conclusion of a $\rightarrow$E, it has to be inferred directly from the consequent of the implication. This means that it either has $\varphi \vee \psi$ as a subformula, which is assumed not to be the case, or leads to it using $\vee$I in one of its subproofs on either φ or ψ. So, φ or ψ can be inferred by substituting all occurrences of $\varphi \vee \psi$ with the respective. □

3 Complexity of Core Logic

In Section 2.4, we saw that if there is a proof of the sequent $\Gamma : \varphi$ in Intuitionistic Logic, then there is a corresponding Core proof of some subsequent of it. Tennant claims that the decision problem of Core Logic is not more complex than the one of Intuitionistic Logic. He also states that Classical Core Logic is not more complex than Classical Logic [Ten92]. We will give a reduction from Core Logic to Intuitionistic Logic in Section 3.1 that can analogously be applied as a reduction from Classical Core to Classical Logic.

In [Göd33] Gödel gives a translation from Intuitionistic Propositional Logic into the Modal Logic **S4**. The equivalence of the two systems was shown in [MT48] by McKinsey and Tarski. Ladner proved the decision problem for the modal logic **S4** to be in **PSPACE** [Lad77]. **PSPACE**-hardness of Intuitionistic Propositional Logic was shown by Richard Statman in [Sta79]. In the Section 3.2, we will adapt the latter proof to Core Logic and reproduce it in a slightly more straightforward way.

3.1 Reduction to Propositional Intuitionistic Logic

For the reduction it is crucial to bear in mind that the decision problem for Core Logic is defined as follows:

Definition 3.1. The decision problem of Core Logic: Given a set of premises Γ and a single sentence φ. Is there a Core proof of $\Delta : \varphi$ or of $\Delta : \bot$ with $\Delta \subseteq \Gamma$?

Theorem 3.2. *The decision problem of Propositional Core Logic is in* **PSPACE**.

Proof. By giving a polynomial time algorithm with oracle access to the decision proplem of Intuitionistic Propositional Logic. The input is a sequent of the form $\Gamma : \varphi$. If the answer to the oracle-question $\Gamma : \bot?$ is positive, then the problem, too, is positively settled. Otherwise, the output is the answer to the question $\Gamma : \varphi?$. □

3.2 Reduction from TQBF

To prove **PSPACE**-hardness we show a reduction from the **PSPACE**-complete problem TQBF (see [Sto76]) to Propositional Core Logic. TQBF refers to the problem of evaluation of quantified Boolean formulas (QBF). Let A be a QBF in prenex normal form $A = Q_n x_n ... Q_1 x_1 B_0$ with $Q_i \in \{\exists, \forall\}$ and B_0 quantifier-free. We will construct a propositional sentence A^* that can be derived from A in polynomial time and is provable in Propositional Intuitionistic Logic (**I**) if and only if A is true.

We define formulas B_{k+1} for $k = 0...n-1$ as $B_{k+1} = Q_{k+1} x_{k+1} B_k$.

The sentence A^+, which is defined below, will be an intermediate step that translates the Boolean quantifiers into Core disjunctions. Its size is exponential in the length of A. Later on, we will build a sentence A^* in polynomial time from A and show that it is Core provable if and only if A^+ is Core provable. Define A^+ as follows:

$$A^+ = B_n^+$$
$$B_0^+ = \neg\neg B_0$$
$$B_{k+1}^+ = (x_{k+1} \vee \neg x_{k+1}) \to B_k^+ \qquad \text{if } Q_{k+1} = \forall$$

and

$$B_{k+1} = (x_{k+1} \to B_k^+) \vee (\neg x_{k+1} \to B_k^+) \qquad \text{if } Q_{k+1} = \exists.$$

For every x_k bound by the $\exists$-Quantifier $Q_k = \exists_j$ there is a function f_j returning an evaluation of x_k that verifies A depending on the evaluation of all $\forall$-quantified variables that have x_k within their scope (see, e.g., [Haz02]). So we have $f_j(I_n, ..., I_{k+1}) = I_k$ where $I_i \in \{0, 1\}$ is an evaluation of x_i.

Lemma 3.3. *A QBF A, in prenex normal form as described above, is true if and only if there is a verifying tree $\mathcal{T}$, with all its nodes representing satisfying assignments of A, that is of the following form:*

1. *The root is* $\emptyset$.
2. *All leaves* Θ *satisfy* B_0.
3. *If* Θ *is a node with height* k *in* $\mathcal{T}$ *and* $Q_k = \exists_j$, *then its child node is* $(\Theta, f_j(\Theta))$.
4. *If* Θ *is a node with height* k *in* $\mathcal{T}$ *and* $Q_k = \forall$, *then its child nodes are* $(\Theta, 0)$ *and* $(\Theta, 1)$.

Proof. "$\Rightarrow$": By induction on the height k.

Base case: A is true, hence the root $\emptyset$ is a satisfying assignment.

Induction hypothesis: All nodes with height $> k$ are satisfying assignments of A.

Induction step: By induction hypothesis, Θ is a satisfying assignment. We have two cases.

1. If $Q_k = \exists_j$, then $f_j(\Theta)$ returns an assignment for the quantified variable x_k that satisfies A. So $(\Theta, f_j(\Theta))$ is a satisfying assignment for A.
2. If $Q_k = \forall$, then both assignments 0 and 1 for the quantified variable x_k satisfy A. So $(\Theta, 0)$ and $(\Theta, 1)$ are satisfying assignments for A.

Accordingly, all leaves (thus complete assignments), satisfy A, so they satisfy B_0.

"$\Leftarrow$": By induction on the height k.

Base case: If there is a verifying tree of the above form, then all leaves satisfy B_0.

Induction hypothesis: All nodes Θ_k of height $\leq k$ are satisfying assignments for B_k.

Induction step: By induction hypothesis, Θ_k is a satisfying assignment for B_k. We have two cases.

1. If $Q_{k+1} = \exists_j$, then Θ_k is a satisfying assignment for B_k, so B_{k+1} is true.
2. If $Q_{k+1} = \forall$, then, by construction, for each node Θ_{k+1} we have two nodes $(\Theta_{k+1}, 0)$ and $(\Theta_{k+1}, 1)$ both satisfying B_k, so B_{k+1} is true.

Accordingly, B_n (i.e., A) is true. □

Lemma 3.4. *The QBF sentence* A *is true if and only if* A^+ *is Core provable.*

In the given proof the following results are used. The Equations 3.1 to 3.4 were proved in Section 2.5 whereas Equation 3.5 represents an application of ∨I in Core Logic.

$$\text{If } \Gamma \nvdash_{\mathbb{C}} \bot, \text{ then } \Gamma \vdash_{\mathbb{C}} \varphi \Leftrightarrow \Gamma \vdash_{\mathbb{C}} \neg\neg\varphi. \tag{3.1}$$

$$\text{If } \Gamma \cup \{\varphi\} \nvdash_{\mathbb{C}} \bot, \text{ then } \Gamma \vdash_{\mathbb{C}} \varphi \to \psi \Leftrightarrow \Gamma \cup \{\varphi\} \vdash_{\mathbb{C}} \psi \tag{3.2}$$

with the "⇐"-direction of Equation 3.2 being an application of the Core rule →I.

$$\text{If } \Gamma \cup \{\varphi\} \nvdash_{\mathbb{C}} \bot \text{ and } \Gamma \cup \{\psi\} \nvdash_{\mathbb{C}} \bot, \text{ then}$$

$$\Gamma \cup \{\varphi \vee \psi\} \vdash_{\mathbb{C}} \theta \Leftrightarrow \Gamma \cup \{\varphi\} \vdash_{\mathbb{C}} \theta \text{ and } \Gamma \cup \{\psi\} \vdash_{\mathbb{C}} \theta. \tag{3.3}$$

$$\text{If } \Gamma \text{ does not contain } \vee, \text{ then } \Gamma \vdash_{\mathbb{C}} \varphi \vee \psi \Rightarrow \Gamma \vdash_{\mathbb{C}} \varphi \text{ or } \Gamma \vdash_{\mathbb{C}} \psi. \tag{3.4}$$

$$\Gamma \vdash_{\mathbb{C}} \varphi \text{ or } \Gamma \vdash_{\mathbb{C}} \psi \Rightarrow \Gamma \vdash_{\mathbb{C}} \varphi \vee \psi. \tag{3.5}$$

Proof. "⇒": Let $A = Q_n x_n ... Q_1 x_1 B_0$ be true. Then, by Lemma 3.3, there is a verifying tree $\mathcal{T}_1$. The nodes are tuples $(b_n, ..., b_k)$ with $b_i \in \{0, 1\}$ encoding satisfying assignments for $x_n, ..., x_k$. Each child node extends the assignment of its parent node by one bit.

We will write l_i for the literals x_i and $\neg x_i$. Let l_i be x_i if the ith bit of Θ is 1 and let l_i be $\neg x_i$ if the ith bit of Θ is 0.

Claim. *If Θ is a node with height k in $\mathcal{T}_1$, then there is a proof of $\{l_n, ..., l_{k+1}\} \vdash_{\mathbb{C}} B_k^+$.*

Proof of claim. By induction on the structure of $\mathcal{T}_1$.

Base: All leaves are total assignments of B_0 that make B_0 true. Hence, B_0 is a classical consequence of $\{l_n, ..., l_1\}$. Every satisfiable set of formulas is consistent. Therefore, by Equation 3.1 $\{l_n, ..., l_1\} \vdash_{\mathbb{C}} \neg\neg B_0$ with $\neg\neg B_0 = B_0^+$.

Induction hypothesis: The above claim holds for all children of the considered node.

Induction step: Consider a node Θ of height k.

1. If $Q_k = \exists_j$, node Θ has only one child $(\Theta, f_j(\Theta))$.

$\overset{\text{IH}}{\Rightarrow} \{l_n, ..., l_k\} \vdash_{\mathbb{C}} B^+_{k-1}$

$\overset{3.2}{\Rightarrow} \{l_n, ..., l_{k+1}\} \vdash_{\mathbb{C}} l_k \rightarrow B^+_{k-1}$

$\overset{3.5}{\Rightarrow} \{l_n, ..., l_{k+1}\} \vdash_{\mathbb{C}} (x_k \rightarrow B^+_{k-1}) \vee (\neg x_k \rightarrow B^+_{k-1})$

with $(x_k \rightarrow B^+_{k-1}) \vee (\neg x_k \rightarrow B^+_{k-1}) = B^+_k$ if $Q_k = \exists$ by construction of A^+

$\Rightarrow \{l_n, ..., l_{k+1}\} \vdash_{\mathbb{C}} B^+_k$

2. If $Q_k = \forall$, node Θ has two children $(\Theta, 0)$ and $(\Theta, 1)$.

$\overset{\text{IH}}{\Rightarrow} \{l_n, ..., x_k\} \vdash_{\mathbb{C}} B^+_{k-1}$ and $\{l_n, ..., \neg x_k\} \vdash_{\mathbb{C}} B^+_{k-1}$

$\overset{3.3}{\Rightarrow} \{l_n, ..., l_{k+1}\} \cup \{(x_k \vee \neg x_k)\} \vdash_{\mathbb{C}} B^+_{k-1}$

$\overset{3.2}{\Rightarrow} \{l_n, ..., l_{k+1}\} \vdash_{\mathbb{C}} (x_k \vee \neg x_k) \rightarrow B^+_{k-1}$

With $(x_k \vee \neg x_k) \rightarrow B^+_{k-1} = B^+_k$ if $Q_k = \forall$ by construction of A^+

$\Rightarrow \{l_n, ..., l_{k+1}\} \vdash_{\mathbb{C}} B^+_k$

The root of the tree is $\emptyset$ and its height is n therefore $\vdash_{\mathbb{C}} B^+_n$, hence $\vdash_{\mathbb{C}} A^+$, is true.

"$\Leftarrow$": Let $\vdash_{\mathbb{C}} A^+$ be true. We construct a tree $\mathcal{T}_2$ from A^+ and show that it is a verifying tree for A. This will be done by applying the Equations (3.2), (3.3) and (3.5) on $\{l_n, ..., l_{k+1}\} \vdash_{\mathbb{C}} B^+_k$ to infer $\{l_n, ..., l_{k+1}, l_k\} \vdash_{\mathbb{C}} B^+_{k-1}$ for $k = n...0$ and take the set of premises of the resulting sequents as our child nodes. Neither x_k nor $\neg x_k$ occurs in $\{l_n, ..., l_{k+1}\}$. Therefore, $\{l_n, ..., l_{k+1}\} \cup x_k$ and $\{l_n, ..., l_{k+1}\} \cup x_k$ are consistent sets and the equations hold.

Construct $\mathcal{T}_2$ as follows:

We have $I_i = 1$ if $l_i = x_i$ and $I_i = 0$ if $l_i = \neg x_i$.

1. The root is $\emptyset$.
2. If $(I_n, ..., I_{k+1})$ is a node in $\mathcal{T}_2$, we have two cases:

a) If $Q_k = \forall$, then $B^+_k = (x_k \vee \neg x_k) \rightarrow B^+_{k-1}$. We can infer

$$\{l_n, ..., l_{k+1}\} \vdash_{\mathbb{C}} (x_k \vee \neg x_k) \rightarrow B^+_{k-1}$$
$$\overset{3.2}{\Rightarrow} \{l_n, ..., l_{k+1}\} \cup \{(x_k \vee \neg x_k)\} \vdash_{\mathbb{C}} B^+_{k-1}$$
$$\overset{3.3}{\Rightarrow} \{l_n, ..., l_{k+1}, x_k\} \vdash_{\mathbb{C}} B^+_{k-1} \text{ and } \{l_n, ..., l_{k+1}, \neg x_k\} \vdash_{\mathbb{C}} B^+_{k-1}$$

and take $(I_n, ..., I_{k+1}, 0)$ and $(I_n, ..., I_{k+1}, 1)$ as child nodes of $(I_n, ..., I_{k+1})$.

b) If $Q_k = \exists$, we have $\{l_n, ..., l_{k+1}\} \vdash_{\mathbb{C}} (x_k \rightarrow B^+_{k-1}) \vee (\neg x_k \rightarrow B^+_{k-1})$. By Equation (3.4), there must be a proof from $\{l_n, ..., l_{k+1}\}$ of one of the disjuncts. So if $\{l_n, ..., l_{k+1}\} \vdash_{\mathbb{C}} (l_k \rightarrow B^+_{k-1})$ is true we infer

$$\overset{3.2}{\Rightarrow} \{l_n, ..., l_{k+1}, l_k\} \vdash_{\mathbb{C}} B^+_{k-1}$$

and take $(I_n, ..., I_{k+1}, I_k)$ as single child node of $(I_n, ..., I_{k+1})$.

Claim. *If $(I_n, ..., I_k)$ occurs in $\mathcal{T}_2$, then it is a satisfying assignment for B_k.*

Proof of claim. By induction on k.

Base: $(I_n, ..., I_1)$ is a leaf of $\mathcal{T}_2$ hence $\{l_n, ..., l_1\} \vdash_{\mathbb{C}} B^+_0$. With $B^+_0 = \neg\neg B_0$ Equation (3.1) gives us $\{l_n, ..., l_1\} \vdash_{\mathrm{C}} B_0$. So the assignment $(I_n, ..., I_1)$ satisfies A.

Induction hypothesis: The above claim holds for all B_i with $0 \leq i \leq k$.

Induction step: There are two cases

1. If $Q_{k+1} = \exists$: $(I_n, ..., I_{k+2})$ appears in $\mathcal{T}_2$, then it has a child $(I_n, ..., I_{k+1})$. That, by induction hypothesis, is a satisfying assignment for B_k. Hence $\exists x_{k+1} B_k$ with assignment $(I_n, ..., I_{k+2})$. Thus, $(I_n, ..., I_{k+2})$ satisfies B_{k+1}.

2. If $Q_{k+1} = \forall$: $(I_n, ..., I_{k+2})$ appears in $\mathcal{T}_2$, then it has two children $(I_n, ..., I_{k+2}, 0)$ and $(I_n, ..., I_{k+2}, 1)$. These, by induction hypothesis, are both satisfying assignments for B_k. Hence $\forall x_{k+1} B_k$ with assignment $(I_n, ..., I_{k+2})$. Thus, $(I_n, ..., I_{k+2})$ satisfies B_{k+1}.

Hence, all nodes $(I_n, ..., I_{k+1})$ are satisfying assignments for B_k.

In particular, $\emptyset$ satisfies B_n. So $\mathcal{T}_2$ is a verifying tree for A as in Lemma 3.3, hence by Lemma 3.3 A is true. $\square$

Because A^+ cannot be obtained from A in polynomial time, we build A^* from A as follows:

Define new variables $y_0, ... y_n$.

$$A^* = B_0^* \rightarrow (B_1^* \rightarrow (...(B_n^* \rightarrow y_n))),$$
$$B_0^* = \neg\neg B_0 \leftrightarrow y_0,$$
$$B_{k+1}^* = ((x_{k+1} \vee \neg x_{k+1}) \rightarrow y_k) \leftrightarrow y_{k+1} \qquad \text{if } Q_{k+1} = \forall$$

and

$$B_{k+1}^* = ((x_{k+1} \rightarrow y_k) \vee (\neg x_{k+1} \rightarrow y_k)) \leftrightarrow y_{k+1} \qquad \text{if } Q_{k+1} = \exists$$

Lemma 3.5. *A^+ is provable in $\mathbb{C}$ if and only if A^* is provable in $\mathbb{C}$.*

Proof. Suppose $\vdash_{\mathbb{C}} A^+$ is true.

Claim. $\{B_0^*, ..., B_n^*\} \vdash_{\mathbb{C}} y_k \leftrightarrow B_k^+$

Proof of claim. By induction on k.

Base: It is easy to see that $B_0^* = B_0^+ \leftrightarrow y_0$. Therefore, $\{B_0^*, ..., B_n^*\} \vdash_{\mathbb{C}} y_0 \leftrightarrow B_0^+$.

Induction hypothesis: The above claim holds for all $y_i \leftrightarrow B_i^+$ with $i \leq k$.

Induction step: Assume $\{B_0^*, ..., B_n^*\}$. We have the two following cases:

1. If $Q_k = \exists$, then $B_{k+1}^* = ((x_{k+1} \rightarrow y_k) \vee (\neg x_{k+1} \rightarrow y_k))$

$$\overset{\text{IH}}{\Rightarrow} B_{k+1}^* = \overbrace{((x_{k+1} \rightarrow B_k^+) \vee (\neg x_{k+1} \rightarrow B_k^+))}^{B_{k+1}^+} \leftrightarrow y_{k+1}$$
$$\Rightarrow B_{k+1}^* = B_{k+1}^+ \leftrightarrow y_{k+1}.$$

2. If $Q_k = \forall$, then $B^*_{k+1} = ((x_{k+1} \vee \neg x_{k+1}) \to y_k) \leftrightarrow y_{k+1}$

$$\overset{\text{IH}}{\Rightarrow} B^*_{k+1} = \overbrace{((x_{k+1} \vee \neg x_{k+1}) \to B^+_k)}^{B^+_{k+1}} \leftrightarrow y_{k+1}$$
$$\Rightarrow B^*_{k+1} = B^+_{k+1} \leftrightarrow y_{k+1}.$$

Thus, $\{B^*_0, \ldots, B^*_n\} \vdash_{\mathbb{C}} y_n \leftrightarrow B^+_n$. B^+_n is provable, so $\{B^*_0, \ldots, B^*_n\} \vdash_{\mathbb{C}} y_n$. Hence, by Equation (3.2) $\vdash_{\mathbb{C}} A^*$.

Now suppose A^* is Core provable. Then, there is a natural deduction proof of y_n from $\{B^*_0, \ldots, B^*_n\}$. Replace any occurrence of y_k for $0 \leq k \leq n$ with B^+_k. The result is a proof of B^+_n from $\{(B^+_0 \leftrightarrow B^+_0), \ldots, (B^+_n \leftrightarrow B^+_n)\}$. Since $A^+ = B^+_n$ we have a proof of A^+ from $\{(B^+_0 \leftrightarrow B^+_0), \ldots, (B^+_n \leftrightarrow B^+_n)\}$, which is obviously Core provable from $\emptyset$. So, $\vdash_{\mathbb{C}} A^+$ is true. □

Theorem 3.6. *Core Logic is* **PSPACE**-*hard.*

Proof. A^* is a polynomial time reduction from TQBF to Propositional Core Logic. TQBF is **PSPACE**-complete, so Propositional Core Logic is **PSPACE**-hard. □

Theorem 3.7. *Core Logic is* **PSPACE**-*complete.*

Proof. Core Logic is in **PSPACE** by Theorem 3.2 and **PSPACE**-hard by Theorem 3.6. Therefore, it is **PSPACE**-complete. □

4 Core Logic in Automated Reasoning

If scientists aim to use machines to help them reason formally, it seems obvious that proofs that make use of EFQ in an uncontrolled manner will not necessarily end up with satisfying results. This leads to the realization that relevance in reasoning is not only a philosophical matter but that it is of practical value in automated reasoning.

One benefit of Core Logic, compared to Intuitionistic Logic, is that it focuses on making proof finding workable for machines that will not distinguish between reasonable, hence acceptable, and arbitrary, hence not acceptable, situations to apply EFQ. One example of such a reasonable EFQ application (in the Intuitionistic Calculus) is Disjunctive Syllogism. Intuitionistic Logic requires to infer the wanted conclusion from the absurdity of one disjunct to continue with the other one. Of course this is something a human reasoner would not notice for they would *know* why they made use of EFQ; yet for a machine it makes no difference. Core Logic allows reasoners not to pay the absurd case any further attention by simply turning to the consistent case; still, they will get to the same conclusion.

Core Logic does not change the relation of consequence. If there is a proof Π of $\perp$ from a set of premises Γ, the answer to the question whether some sentence φ follows from Γ is positive; even in Core Logic. There simply is no concealment of the manner how this answer is reached. In other words: The automated proof finder based on Core rules would return the proof of $\Gamma : \perp$ and the *user*, a human logician, could feel free to add a step of EFQ to infer φ. This makes the Core result at least just as expressive as any result a proof finder based on the Intuitionistic Calculus would output. It is conceivably even better because logicians *know* about the irrelevance; they added it all by themselves. As a matter of fact, all Tennant has done, albeit in a thorough way, is to restrict how proofs are constructed properly to eschew *hidden* fallacies of relevance within them. In that way, an automated, hence strictly structural, proof finder is forced to stay *on topic*. This should reassure any hypothetical user of automated proof search.

In Section 2.4, we saw that it is possible to find a Core proof for an arbitrary Intuitionistic proof. So, Tennant managed to reformulate the rules of the established calculi in a way that the rule of EFQ is not needed for the completeness of the deduction system. Even $\varphi, \neg\varphi \vDash \psi$ is not rejected by this claim because it has a subsequent $\varphi, \neg\varphi \vdash_{\mathbb{C}} \bot$ that is provable in Core Logic.

Minimal Logic in Automated Reasoning As mentioned in Subsection 1.2.2, Minimal Logic does not make use of EFQ and its propositional fragment is in **PSPACE**, as is Propositional Intuitionistic Logic. Still, if the logical system that a proof search algorithm executes is supposed to uncover inconsistencies to eschew irrelevances, Minimal Logic cannot be the system of choice considering automated proof search. The reason is the lack of restriction regarding the discharge rules, which involves the possibility to infer an arbitrary negation from a contradiction. There is no benefit compared with other logical systems that make use of EFQ regarding the irrelevance that could be created. Also, in contrast to Core Logic, Minimal Logic does not proof Disjunctive Syllogism, which is a well accepted logical principle often used in mathematical reasoning.

Relevance Logic in Automated Reasoning As we have seen, relevance in computational reasoning can be valuable to get adequate results. The Relevance Logic **R** approaches this issue by relevantizing the conditional. This changes the well accepted truth table of this connective and requires new semantics to go with it. Core Logic, on the other hand, offers a propositional calculus that does not change the meanings of the connectives but restricts the deducibility relation.

> *The Core logician relevantizes, however, at the 'level of the turnstile', and not by dramatically altering the logical behavior of the object language conditional* $\rightarrow$. ([Ten17, p. 263])

This is the reason why Propositional Core Logic is decidable. All intuitionistic theorems are Core theorems and vice versa [Ten17, p. 263]. In contrast, Propositional

Relevance Logic as formulated by Anderson and Belnap has a decision problem that is not decidable (see [Urq84]) and its decidable fragment has a decision problem that is *at best* **ESPACE**-hard (see [Urq90]), with **ESPACE** being the class of decision problems that are solvable by a deterministic Turing machine in SPACE($2^{O(n)}$). So **R** cannot be of any practical importance in automated reasoning.

5 Conclusion

Core Logic is not a significantly different logical system. It is a slight modification of the established system Minimal Logic. Core Logic (resp. Classical Core Logic) enables Intuitionists (resp. Classicists) to infer—from a consistent set of premises—anything that is deducible in the respective non-relevant system. The transitivity of deducibility is ensured for consistent combinations of sets of premises, whereas if the set of combined premises allows the derivation of $\bot$, the $\bot$ will mark the preceding deduction as a dead end. No truth lies in it and its persuasion would be fruitless. This seems reasonable regarding the way a human mind would probably revise the assumptions that were made in the first place, if they prove inconsistent, instead of inferring whatever one pleases and continuing with the proof.

Tennant gives a set of inference rules that do not change the generally accepted meanings of logical operators but still adjust the deducibility relation. He forces relevance by restricting the discharge rules and relaxing the rules of $\vee$E and $\rightarrow$I. The latter modifications are needed to eschew the necessity of EFQ. His efforts lead to a relevance property as described in Section 2.1.

Consequently, the decision problem for Core Logic has been defined differently than for Intuitionistic or Classical Logic. Core logicians do not search for a proof of some given sentence φ but for a proof of some subset of $\{\varphi\}$ to preserve relevance and become aware of any inconsistency in their assumptions. As discussed in Section 4, this does not affect the relation of semantical consequence.

In the Chapter 3, we saw that the theorem proving problem for Core Logic is not more complex than the theorem proving problem for Intuitionistic Logic. Moreover, the fact that a proof search could be aborted once a deduction of absurdity is encountered seems quite reassuring that proof search algorithms could actually be faster in practice.

Our main result is the affirmation of **PSPACE**-completeness for Core Logic. We saw how Statman's **PSPACE**-hardness proof for Propositional Intuitionistic Logic applies to Core Logic. So, we conclude that the computational complexity is not affected for the worse but unfortunately, neither for the better by relevantizing Intuitionistic Logic in Tennant's notion.

The main achievement of Tennant can be summarized as defining rules that output just as expressive results as the established ones without making use of EFQ and still not increasing the computational complexity. Regarding automated proof finding, Core Logic could be a practicable possibility to implement paraconsistency in complex systems that are likely to produce inconsistencies. It is practicable because of the moderate computational complexity, compared to other relevance logics, in the propositional case.

In a subsequent work, an examination of the actual practicability of (Classical) Core Logic could confirm Tennant's hope of even faster proof finding than with present proof finders [Ten92, p. 8]. Since Artificial Intelligence techniques make use of automated theorem proving, it would be of great interest for computer scientists to inspect the applicability of Core Logic for these fields. This should result in determining what impact Core Logic could actually make in contemporary and future technologies.

Bibliography

[AB75] Alan Ross Anderson, Nuel D. Belnap. *Entailment: The Logic of Relevance and Necessity. Vol I.* Princeton, NJ: Princeton University Press, 1975.

[Gen35] Gerhard Gentzen. *Untersuchungen über das logische Schließen. I*. Mathematische Zeitschrift **39** (1935), no. 1, pp. 176–210. DOI: 10.1007/BF01201353.

[Göd33] Kurt Gödel. *Eine Interpretation des Intuitionistischen Aussagenkalküls*. Ergebnisse eines mathematisches Kolloquiums **4** (1933), pp. 39–40.

[Haz02] Michiel Hazewinkel. *Encyclopaedia of Mathematics*. Berlin and New York: Springer-Verlag, 2002.

[Joh37] Ingebrigt Johansson. *Der Minimalkalkül, ein reduzierter intuitionistischer Formalismus*. Compositio Mathematica **4** (1937), pp. 119–136.

[Kri63] Saul A. Kripke. *Semantical Analysis of Intuitionistic Logic I*. Formal Systems and Recursive Functions: Proceedings of the Eighth Logic Colloquium, Oxford July 1963. Ed. by Michael Dummett, J. N. Crossley. North Holland, 1963, pp. 92–130.

[Lad77] Richard E Ladner. *The Computational Complexity of Provability in Systems of Modal Propositional Logic*. SIAM Journal on Computing **6** (1977), no. 3, pp. 467–480.

[Mar14] Edwin Mares. *Relevance Logic*. The Stanford Encyclopedia of Philosophy. Ed. by Edward N. Zalta. Spring 2014. Metaphysics Research Lab, Stanford University, 2014.

[MT48] John CC McKinsey, Alfred Tarski. *Some Theorems About the Sentential Calculi of Lewis and Heyting*. The journal of symbolic logic **13** (1948), no. 1, pp. 1–15.

[Raa18] Panu Raatikainen. *Gödel's Incompleteness Theorems*. The Stanford Encyclopedia of Philosophy. Ed. by Edward N. Zalta. Fall 2018. Metaphysics Research Lab, Stanford University, 2018.

[Rea88] Stephen Read. *Relevant Logic: a Philosophical Examination of Inference*. (1988).

[Sta79] Richard Statman. *Intuitionistic Propositional Logic is Polynomial-Space Complete*. Theoretical Computer Science **9** (1979), no. 1, pp. 67–72. DOI: 10.1016/0304-3975(79)90006-9.

[Sto76] Larry J. Stockmeyer. *The Polynomial-Time Hierarchy*. Theoretical Computer Science **3** (1976), no. 1, pp. 1–22. DOI: 10.1016/0304-3975(76)90061-X.

[Ten05] Neil Tennant. *Relevance in Reasoning*. The Oxford handbook of philosophy of mathematics and logic. Ed. by Stewart Shapiro. Oxford handbooks in philosophy. Oxford: Oxford University Press, 2005, pp. 696–726. DOI: 10.1093/0195148770.003.0023.

[Ten12] Neil Tennant. *Cut for Core Logic*. The Review of Symbolic Logic **5** (2012), no. 03, pp. 450–479. DOI: 10.1017/S1755020311000360.

[Ten17] Neil Tennant. *Core logic*. First edition. Oxford United Kingdom: Oxford University Press, 2017.

[Ten78] N. Tennant. *Entailment and Proofs*. Proceedings of the Aristotelian Society **79** (1978), pp. 167–viii.

[Ten92] Neil Tennant. *Autologic*. Vol. 9. Edits. Edinburgh: Edinburgh University Press, 1992.

[Urq84] Alasdair Urquhart. *The Undecidability of Entailment and Relevant Implication*. The Journal of Symbolic Logic **49** (1984), no. 4, pp. 1059–1073.

[Urq90] Alasdair Urquhart. *The Complexity of Decision Procedures in Relevance Logic*. Truth or consequences. Springer, 1990, pp. 61–76.

Erklärung der Selbstständigkeit

Hiermit versichere ich, die vorliegende Masterarbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel verwendet zu haben. Die Arbeit hat in gleicher oder ähnlicher Form noch keinem anderen Prüfungsamt vorgelegen.

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