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Masterarbeit

Modal Team Logic

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1 Introduction

Modal logic is a popular way to describe and check various properties of a system in example like a integrated circuit. The mostly used modal logic, for this purpose, is the computational tree logic (CTL). But complex systems often have dependencies between their properties/variables and to express these dependencies is very difficult in CTL. To model these dependence between properties, in first-order logic, the concept of functional dependence was introduced by Väänänen in [Vää07]. Functional dependence means that a property p_n depends on $p_1, \ldots, p_{n-1}$, if a function f exists which defines p_n by $f(p_1, \ldots, p_{n-1})$, this is denoted by $dep(p_1, \ldots, p_{n-1}; p_n)$.
To decide if a property depends on other properties we need the concept of teams, because each member of a team has its own property values. A property p_n only depends on properties $p_1, \ldots, p_{n-1}$ relative to a team T, if and only if $p_n = f(p_1, \ldots, p_n$ holds on every team member. We can interpret a team as states or points of time of our system. In the following example we a have system with four properties $p_1, \ldots, p_4$ and we measure the values of these properties to three different points of time.

Time	p_1	p_2	p_3	p_4
0	0	1	0	0
1	0	0	1	1
3	1	1	0	0

Table 1.1: Snapshot's of a system with four properties

In this example we easily can see that the property p_3 only depends on the property p_2 or p_4, but property p_1 has no effect to p_3.
Modal dependence logic (MDL) integrates these concepts of dependence into modal logic and was introduced by Väänänen in [Vää08]. In modal dependence logic a team is a set of worlds of a Kripke model, and property values are the labelings of those worlds.
In this master thesis we will study an extension of MDL the modal team logic (MTL).In MDL classic semantical negation isn't allowed, this causes that we cannot express that a dependence of properties doesn't hold, but in MTL this operation is allowed.
In [EL11] Ebbing and Lohmann study the model checking problem of modal dependence logic. The main result is, that model checking in modal dependence logic is NP-complete. In this thesis we will also study the model checking problem of modal team logic to compare the results and look at how the classical syntactical negation effects the complexity.

2 Preliminaries

Before we classify the complexity of the *MTL* model checking problem, we will define modal team logic analogous to modal logic ([BdRV01]) with the concept of dependence and the classic syntactical negation. Furthermore we define the polynomial hierarchy so that we can classify the *MTL* model checking Problem above of *NP*.

2.1 Modal Team Logic

2.1.1 Syntax

Let *Var* be a set of atomic propositions and $p \in Var$, then the set of *MTL* formulas is defined by the following grammar:

$$\varphi \underset{\mathsf{def}}{=} \top \mid \bot \mid p \mid \bar{(}p) \mid dep(p_1, \ldots p_{n-1}; p_n) \mid \neg dep(p_1, \ldots p_{n-1}; p_n) \mid$$
$$\theta_1 \wedge \theta_2 \mid \theta_1 \otimes \theta_2 \mid \theta_1 \vee \theta_2 \mid {\sim}\theta \mid \Box\theta \mid \Diamond\theta$$

Definition 2.1.1 (Negation depth).
The negation depth $\deg_{\sim}(\theta)$ *is inductively defined as:*

$$\begin{aligned}
\deg_{\sim}(p) &\underset{\mathsf{def}}{=} \deg_{\sim}(\overline{p}) \underset{\mathsf{def}}{=} 0 && , p \in Var \\
\deg_{\sim}(\theta_1 * \theta_2) &\underset{\mathsf{def}}{=} max(\deg_{\sim}(\theta_1), \deg_{\sim}(\theta_2)) && , * \in \{\wedge, \otimes, \vee\} \\
\deg_{\sim}(\Lambda\theta) &\underset{\mathsf{def}}{=} \deg_{\sim}(\theta) && , \Lambda \in \{\Box, \Diamond\} \\
\deg_{\sim}({\sim}\theta) &\underset{\mathsf{def}}{=} 1 + \deg_{\sim}(\theta)
\end{aligned}$$

In some cases, we denote the allowed negation depth k of an *MTL* Logic, with the operator $\sim_k$.

2.1.2 Semantics

In modal logic the validation of a formula is checked in a world of a Kripke model. Validation on *MTL* formulas is defined similar, but *MTL* extends the validation in a world with the concept of teams. A specific *MTL* formula is checked in a set of worlds on a Kripke model.

Definition 2.1.2 (Kripke model).
A Kripke model is a tuple $\mathcal{M} = (W, R, \pi)$ where W is a non-empty set of worlds, π a labeling function on those worlds ($\pi : W \mapsto \mathcal{P}(Var)$) and R a binary relation between worlds.

Definition 2.1.3 (Team).
A team T is a subset of worlds W of a Kripke model $\mathcal{M} = (W, R, \pi)$.

For any team T of a Kripke model $\mathcal{M} = (W, R, \pi)$ the set of successor worlds is defined as follows:

$$R(T) = \{w \in W \mid \exists w' \in T, \text{ s.t. } w'Rw\}$$

Now the set of successor teams is defined as follows:

$$\langle T \rangle_R = \{T' \mid T' \subseteq R(T) \text{ and } \forall w \in T, \text{ there is } w' \in T' \text{ with } (w, w') \in R\}$$

Definition 2.1.4 (Truth evaluation).
The truth of a MTL formula on a team T and a Kripke model $\mathcal{M} = (W, R, \pi)$, denoted as $\mathcal{M}, T \models \varphi$, is defined as follows:

$\mathcal{M}, T \models \top$		*always holds,*
$\mathcal{M}, T \models \bot$	*iff*	$T = \emptyset$,
$\mathcal{M}, T \models p$	*iff*	$p \in \pi(w)$ *for all* $w \in T$,
$\mathcal{M}, T \models \overline{p}$	*iff*	$p \notin \pi(w)$ *for all* $w \in T$,
$\mathcal{M}, T \models dep(p_1, \ldots p_{n-1}; p_n)$	*iff*	*for all* $w_1, w_2 \in T$ *with* $\pi(w_1) \cap \{p_1, \ldots, p_{n-1}\} = \pi(w_2) \cap \{p_1, \ldots, p_{n-1}\}$: $p_n \in \pi(w_1)$ *iff* $p_n \in \pi(w_2)$,
$\mathcal{M}, T \models \neg dep(p_1, \ldots p_{n-1}; p_n)$	*iff*	$T = \emptyset$,
$\mathcal{M}, T \models {\sim}\theta$	*iff*	$\mathcal{M}, T \not\models \theta$ *for* $T \neq \emptyset$,
$\mathcal{M}, T \models \theta_1 \wedge \theta_2$	*iff*	$\mathcal{M}, T \models \theta_1$ *and* $\mathcal{M}, T \models \theta_2$,
$\mathcal{M}, T \models \theta_1 \varovee \theta_2$	*iff*	$\mathcal{M}, T \models \theta_1$ *or* $\mathcal{M}, T \models \theta_2$,
$\mathcal{M}, T \models \theta_1 \vee \theta_2$	*iff*	*there exists sets* T_1, T_2 *with* $T = T_1 \cup T_2$, *s.t.* $\mathcal{M}, T \models \theta_1$ *and* $\mathcal{M}, T \models \theta_1$,
$\mathcal{M}, T \models \Diamond\theta$	*iff*	*there exists a set* $T' \in \langle T \rangle$, *s.t.* $\mathcal{M}, T' \models \theta$,
$\mathcal{M}, T \models \Box\theta$	*iff*	$\mathcal{M}, \{w' \mid \exists w \in T \text{ with } (w, w') \in R\} \models \theta$
$\mathcal{M}, \emptyset \models \theta$		*always holds.*

Definition 2.1.5 (Negation normal form).
Let ψ be a propositional logic formula. Then ψ is in negation normal form, if negation only occurs at atomic propositions.

Corollary 2.1.6.
Every propositional logic formula ψ can be transformed into a formula ψ' which is in negation normal form.

2.2 Model Checking

Definition 2.2.1 (Model-Checking).
Given a Kripke model $\mathcal{M} = (W, R, \pi)$ *and a MTL formula* ψ*. Then the set of all satisfying initial teams is defined as:*

$$M_\alpha = \{T \in \mathscr{P}(W) \mid \mathcal{M}, T \models \psi\}.$$

Model checking is now defined as the question if a team T *over* W *is in the model checking set of* $\mathcal{M}$ *and* ψ*.*

Example 2.2.2 (Model-Checking). *To illustrate model checking in modal team logic we want to create a Kripke model about the situation how rainbows can occur. This is done by the Kripke model* $\mathcal{M}$ *which is shown in figure* 2.1*.*

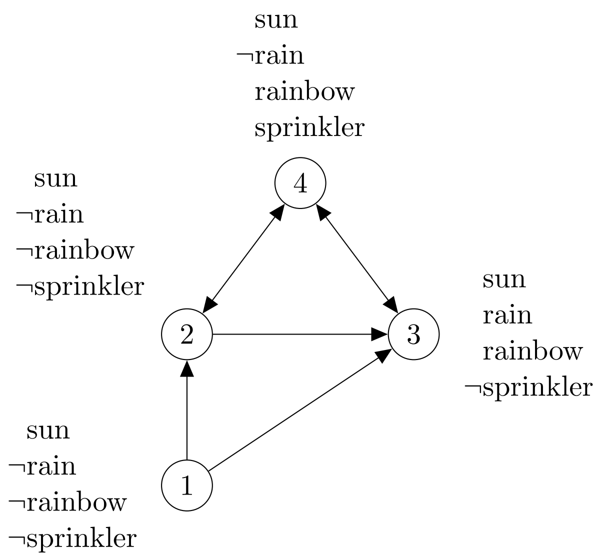


Figure 2.1: Kripke model for the rainbow example

Now if we check the following formula in world 1 *on the Kripke model* $\mathcal{M}$*, it holds that it always holds that the rainbow only depends on the sun and if its raining.*

$$\mathcal{M}, \{1\} \models \Box dep(sun, rain; rainbow)$$

But if we check the formula on the Team $\{1, 3\}$ *we see that this isn't always the case, because in world* 4 *a rainbow occurs, because the sprinkler is active instead of the rain.*

$$\mathcal{M}, \{1, 3\} \not\models \Box dep(sun, rain; rainbow)$$

We can describe this, as that we can find a successor team of $\{1,3\}$ *where the dependence doesn't hold.*

$$\mathcal{M}, \{1, 3\} \models \Diamond{\sim} dep(sun, rain; rainbow)$$

This is possible, because the dependence doesn't hold on the successor team $\{2, 4\}$*.*

2.3 Modal Dependence Logic

Definition 2.3.1 (Modal dependence Logic *MDL*).
Modal dependence logic is similarly defined as modal team logic, but the semantical negation operator $\sim$ isn't in the operator set of MDL.

Definition 2.3.2 (Modal intuitionistic dependence Logic *MIDL*).
Modal intuitionistic dependence Logic is an extension of MDL . MIDL extends MDL by the intuitionistic implication operator $\rightarrow$.
Let $\varphi = \varphi_1 \rightarrow \varphi_2$ be a MIDL implication formula, then φ is valid on the Kripke model $\mathcal{M}$ and the team T, if and only if for all subsets T' of T the following holds:

$$\mathcal{M}, T' \not\models \varphi_1 \text{ or } \mathcal{M}, T' \models \varphi_2$$

2.4 Polynomial Hierarchy

Let $k \in \mathbb{N}$. Then the classes of the polynomial hierarchy are defined as:

$$\begin{aligned}
\Sigma_0^p = \Pi_0^p = \Delta_0^p &\underset{\text{def}}{=} P \\
\Sigma_{k+1}^p &\underset{\text{def}}{=} NP^{\Sigma_k^p} \\
\Pi_{k+1}^p &\underset{\text{def}}{=} \{A \mid \overline{A} \in \Sigma_{k+1}^p\} \\
\Delta_{k+1}^p &\underset{\text{def}}{=} P^{\Sigma_k^p} \\
PH &\underset{\text{def}}{=} \bigcup_{k \geq 0} \Sigma_k^p \cup \Pi_k^p \cup \Delta_k^p
\end{aligned}$$

Furthermore we can restrict the number of oracle questions which a turing machine can ask. This is denoted by adding $[f(n)]$ to the oracle's class. For example $P^{NP[1]}$ is a polynomial time turing machine, which can ask exactly one oracle question to a NP turing machine.

From these definitions we can easily derive some inclusion relations between these classes. Figure 2.2 illustrates these inclusion relations.

$$\Sigma_k^p \cup \Pi_k^p \subseteq \Delta_{k+1}^p \subseteq \Sigma_{k+1}^p \cup \Pi_{k+1}^p$$

Problem (Quantified Boolean formula (QBF))
The quantified Boolean formulas are inductively defined as follows:

IB: $0, 1, x \in \mathit{Var}$ are QBF formulas.

IS:
- $\varphi \in QBF \Rightarrow \neg\varphi \in QBF$
- $\varphi_1, \varphi_2 \in QBF \Rightarrow (\varphi_1 \wedge \varphi_2)$ and $(\varphi_1 \vee \varphi_2) \in QBF$

- $\varphi \in QBF, x \in Var \Rightarrow \exists x\varphi$ and $\forall x\varphi \in QBF$

Theorem 2.4.1 (([Sto77])).
QBF is PSPACE-complete under $\leq_m^p$ reductions.

Definition 2.4.2 (QBF_k).
Let $k \geq 1$, then QBF_k is defined as:

$$QBF_k \underset{\text{def}}{=} \{\psi \mid \psi \textit{ is fully quantified and formed like}$$
$$\exists x_{11}\exists x_{12}\ldots\exists x_{1i_1}\forall x_{21}\forall x_{22}\ldots\forall x_{2i_2}\ldots\partial x_{k1}\partial x_{k2}\ldots\partial x_{ki_k}\psi'$$
$$\textit{where } \psi' \textit{ is a propositional logic formula and } \psi \equiv 1 \textit{ holds.}\}$$

Theorem 2.4.3 (([Wra77])).
QBF_k is Σ_k^p-complete under $\leq_m^p$ reductions.

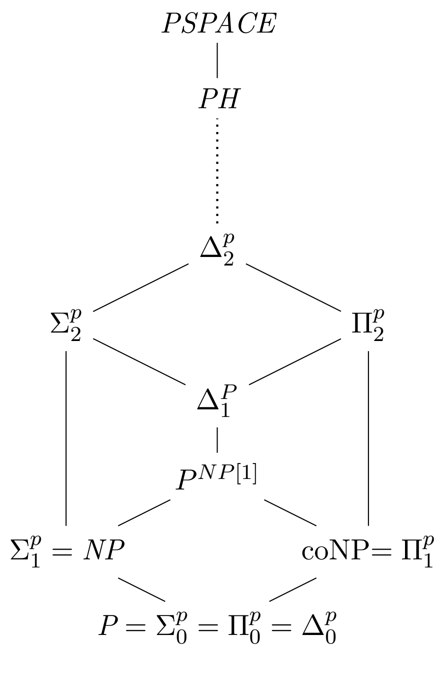


Figure 2.2: Polynomial hierarchy

3 Properties of MTL

In this chapter we study how the presence of the semantical negation "$\sim$" influences the expressivity of *MTL* with respect to *MDL*. Furthermore with the concept of validation on teams, the question arises how the teams are closed under special closure properties.

3.1 Operators

MTL is more expressive compared to modal dependence logic (*MDL*), because with the $\sim$ operator universal quantification can be expressed. In the following some of these universal quantification operators will be defined.

Definition 3.1.1 (Δ Operator).
Δ is a unary modal operator defined as:

$$\Delta\varphi \underset{\text{def}}{=} \sim\Diamond\sim\varphi$$

The Δ operator expresses an universal quantification on successor teams. $\Delta\varphi$ is only satisfied on a Kripke model $\mathcal{M}$ and a team T if and only if for all successor Teams T' it holds that $\mathcal{M}, T' \models \varphi$.

Definition 3.1.2 (⊘ Operator).
⊘ is a binary operator defined as:

$$\varphi_1 \oslash \varphi_2 \underset{\text{def}}{=} \sim(\sim\varphi_1 \vee \sim\varphi_2)$$

Corollary 3.1.3 (Simulation of *MIDL*).
Let φ be a MIDL formula, then there exists an equivalent MTL formula.

Proof. Let $\mathcal{M} = (W, R, \pi)$ be a Kripke model and $T \subseteq W$ a team. Only the implication operator has to be simulated by a *MTL* formula , because all other *MIDL* operators are existing in *MTL*. Let $\varphi = \varphi_1 \rightarrow \varphi_2$ be a *MIDL* implication formula, then φ is valid on the Kripke model $\mathcal{M}$ and the team T, if and only if for all subsets T' of T the following holds:

$$\mathcal{M}, T' \not\models \varphi_1 \text{ or } \mathcal{M}, T' \models \varphi_2$$

This formula can be expressed with *MTL* as follows:

$$\begin{aligned}
\mathcal{M}, T \models \sim((\varphi_1 \ \wedge \sim\varphi_2) \vee \top) \quad & \Leftrightarrow \forall T_1, T_2 \text{ with } T_1 \cup T_2 = T \text{ it holds that} \\
& \quad \mathcal{M}, T_1 \not\models (\varphi_1 \wedge \sim\varphi_2) \text{ or } \underbrace{\mathcal{M}, T_2 \not\models \top}_{\text{always false}} \\
& \Leftrightarrow \forall T_1, T_2 \text{ with } T_1 \cup T_2 = T \text{ it holds that} \\
& \quad \mathcal{M}, T_1 \not\models (\varphi_1 \wedge \sim\varphi_2) \\
& \Leftrightarrow \forall T_1 \subseteq T \text{ it holds that} \\
& \quad \mathcal{M}, T_1 \not\models \varphi_1 \text{ or } \mathcal{M}, T_1 \models \varphi_2 \\
& \Leftrightarrow \mathcal{M}, T \models \varphi_1 \rightarrow \varphi_2
\end{aligned}$$

□

Definition 3.1.4 (→ Operator).
→ is a binary operator defined as:

$$\varphi_1 \rightarrow \varphi_2 \underset{\mathsf{def}}{=} \sim((\varphi_1 \ \wedge \sim\varphi_2) \vee \top)$$

With the → operator an universal quantification on teams can be expressed. On these teams, the operator is similary defined as for propositional implication. $\varphi_1 \rightarrow \varphi_2$ is satisfied by a Kripke model $\mathcal{M}$ and a team T if and only if for all sub teams T' it holds that $\mathcal{M}, T' \not\models \varphi_1$ or $\mathcal{M}, T', \models \varphi_2$

Lemma 3.1.5 (□ is self-dual).
Let φ a MTL formula, then $\Box\varphi$ is equivalent to $\sim\Box\sim\varphi$.

Proof. Let $\theta = \sim\Box\sim\varphi$. θ is satisfied by an Kripke model $\mathcal{M}$ and a team T iff

$$\begin{aligned}
& \mathcal{M}, T \not\models \Box\sim\varphi \\
\Leftrightarrow \quad & \mathcal{M}, R(T) \not\models \sim\varphi \\
\Leftrightarrow \quad & \mathcal{M}, R(T) \models \varphi \\
\Leftrightarrow \quad & \mathcal{M}, T \models \Box\varphi
\end{aligned}$$

□

3.2 Closure properties

Definition 3.2.1 (Downwards closure).
A modal logic has the downwards closure *property if for any formula φ with $\mathcal{M}, T \models \varphi$ and for all $T' \subseteq T$ it holds that $\mathcal{M}, T' \models \varphi$*

Definition 3.2.2 (Upwards closure).
A modal logic has the upwards closure *property if for any formula φ with $\mathcal{M}, T \models \varphi$ and for all $T' \subseteq W$ it holds that $\mathcal{M}, T \cup T' \models \varphi$*

Definition 3.2.3 (Union Closure).
A modal logic has the union closure *property if for any formula φ with $\mathcal{M}, T_1 \models \varphi$ and $\mathcal{M}, T_2 \models \varphi$, it holds that $\mathcal{M}, T_1 \cup T_2 \models \varphi$*

Definition 3.2.4 (Intersection Closure).
A modal logic has the intersection closure *property if for any formula φ with $\mathcal{M}, T_1 \models \varphi$ and $\mathcal{M}, T_2 \models \varphi$, it holds that $\mathcal{M}, T_1 \cap T_2 \models \varphi$*

Lemma 3.2.5.
MTL doesn't have the upwards closure property.

Proof. Let $\mathcal{M} = (W, R, \pi)$ as shown in figure 3.1. Now by contradiction we will show that *MTL* doesn't has the upwards closure property. For team $T_1 = \{w_0, w_1\}$ it holds that

$$\mathcal{M}, T_1 \models {\sim}\Box({\sim}p \wedge {\sim}\overline{p}).$$

Now with $w_2 \in W$ and $T = T_1 \cup \{w_2\}$ it holds that

$$\mathcal{M}, T \models \Box({\sim}p \wedge {\sim}\overline{p}).$$

□

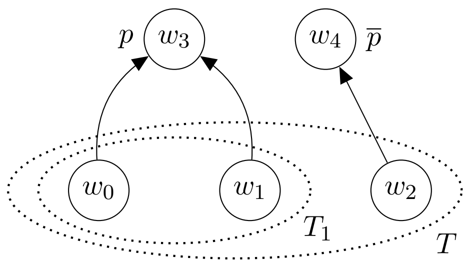


Figure 3.1: Counter example for upwards closure

Lemma 3.2.6.
MTL doesn't have the downwards closure property.

Proof. Let $\mathcal{M} = (W, R, \pi)$ as shown in figure 3.2.
For team $T = \{w_1, w_2\}$ it holds that

$$\mathcal{M}, T \models {\sim}p \wedge {\sim}\overline{p}.$$

By contradiction we assume, that the downward closure property holds. Then the teams $T_1 = \{w_1\}, T_2 = \{w_2\}$ have to satisfy ${\sim}p \wedge {\sim}\overline{p}$. But T_1 satisfies $\overline{p}$, which leads to contradiction with

$$\mathcal{M}, T_1 \models {\sim}p \wedge {\sim}\overline{p}.$$

□

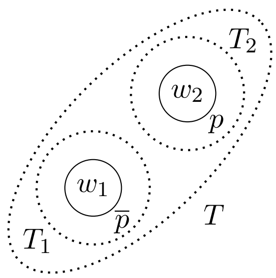


Figure 3.2: Counter example for downwards and union closure

Lemma 3.2.7.
MTL doesn't have the union closure property.

Proof. Let $\mathcal{M} = (W, R, \pi)$ as shown in figure 3.2. For $T_1 = w_1$ and $T_2 = w_2$ it holds that

$$\mathcal{M}, T_1 \models dep(p), \quad \mathcal{M}, T_2 \models dep(p).$$

Contrary for $T = T_1 \cup T_2 = \{w_1, w_2\}$ it holds that

$$\mathcal{M}, T \not\models dep(p).$$

□

Lemma 3.2.8.
MTL doesn't have the intersection closure property.

Proof. Let $\mathcal{M} = (W, R, \pi)$ as shown in figure 3.3. For teams $T1 = \{w_0, w_1\}$ and $T2 = \{w_0, w_2\}$ it holds that

$$\mathcal{M}, T_1 \models {\sim}p \wedge {\sim}\overline{p}$$

$$\mathcal{M}, T_2 \models {\sim}p \wedge {\sim}\overline{p}$$

If we assume that *MTL* has the intersection closure property the following must hold:

$$\mathcal{M}, T_1 \cap T_2 \models {\sim}p \wedge {\sim}\overline{p}$$

But with $T = T_1 \cap T_2 = \{w_0\}$ the following holds

$$\mathcal{M}, T \not\models {\sim}p \wedge {\sim}\overline{p},$$

which leads to a contradiction. □

Lemma 3.2.9 (Downwards closure of $\boldsymbol{\Delta}$ operator).
If φ has the downwards closure property, then the formula $\theta = \Delta\varphi$ has the downwards closure property.

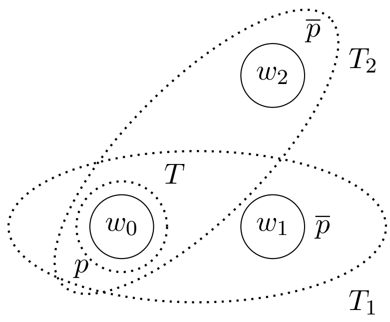


Figure 3.3: Counter example for intersection closure

Proof. Let $\theta = \Delta\varphi$ a *MTL* formula and $\mathcal{M}$ a Kripke model. By contradiction we assume that φ has the downwards closure property and θ doesn't.
By definition $\Delta\varphi$ is satisfied on team T if and only if for all $T' \in \langle T \rangle$, it holds that T' satisfies φ.
If $\Delta\varphi$ isn't downwards closed then for $T^* \subseteq T$ there exists an $U \in \langle T^* \rangle$ with

$$\mathcal{M}, U \not\models \varphi.$$

With $T^* \subseteq T$ it holds that every $U \in \langle T^* \rangle$ is a subset of an element in $\langle T \rangle$. Furthermore with $\mathcal{M}, T' \models \varphi$ and the downwards closure of φ it holds for every subset T'' of T' that

$$\mathcal{M}, T'' \models \varphi.$$

Now there can't exist such an $U \in \langle T^* \rangle$, which doesn't satisfy φ. But this is an contradiction to the assumption that $\Delta\varphi$ is not downwards closed. □

Corollary 3.2.10.
Let φ be a downwards closed MTL formula, then $\Delta\varphi$ is equivalent to $\Box\varphi$.

Proof. $\Delta\varphi$ is satisfied on team T, if and only if $\forall T' \in \langle T \rangle$

$$\mathcal{M}, T' \models \varphi.$$

By the downwards closure property it fulfills to show that $\Delta\varphi$ is satisfied by the biggest team in $\langle T \rangle$, which is $R(T)$. By definition 2.1.4 the assumption holds.

$$\mathcal{M}, R(T) \models \varphi$$

□

4 Complexity of Model Checking

To classify the complexity of MTL-model checking, we study subsets of all MTL operators. In Table 4.1 all model checking results are shown, where Theorem 4.3.1 is the main result of this chapter.

With +,− the presence or absence of an operator is denoted and $*$ means that the operator isn't important for this special complexity result.

$\Box$	$\Diamond$	$\wedge$	$\vee$	$\otimes$	$\overline{\cdot}$	$\sim$	$\sim_k$	dep	Complexity	Proof
$*$	+	+	$*$	$*$	$*$	+	$*$	$*$	$PSPACE$-Complete	Theorem 4.3.1
$*$	$*$	+	+	$*$	$*$	+	$*$	$*$	$PSPACE$-Complete	Theorem 4.3.3
$*$	+	+	$*$	$*$	$*$	−	+	+	Σ^p_{k+1}-Complete	Corollary 4.4.1
$*$	+	−	−	−	$*$	+	$*$	+	$P^{NP[1]}$-Complete	Theorem 4.4.3
$*$	−	−	+	−	$*$	+	$*$	−	P	Corollary 4.4.4
$*$	+	−	−	−	$*$	+	$*$	−	P	Corollary 4.4.5
−	−	−	+	−	$*$	+	−	$*$	open	open

Table 4.1: Complexity results for model checking

4.1 Boolean formula transformations

In this section, we introduce two techniques to reduce the canonical Boolean formula evaluation pattern to the modal team logic model checking problem. In the reductions of the next chapter, this reductions are needed to encode a Boolean assignment into a Kripke structure and a team.

Definition 4.1.1 ($f^{\sim}$ transformation).
For every propositional logic formula φ in negation normal form, the transformation $f^{\sim}(\varphi)$ is inductively defined as:

$$\begin{aligned} p &\mapsto \sim\overline{p}^1 \wedge \overline{p}^0, \\ \overline{p} &\mapsto \overline{p}^1 \wedge \sim\overline{p}^0, \\ \theta_1 \vee \theta_2 &\mapsto f^{\sim}(\theta_1) \otimes f^{\sim}(\theta_2), \\ \theta_1 \wedge \theta_2 &\mapsto f^{\sim}(\theta_1) \wedge f^{\sim}(\theta_2). \end{aligned}$$

Lemma 4.1.2.
Let ψ be a 3CNF Boolean formula in negation normal form, and $\mathcal{M} = (W, R, \pi)$ be a Kripke model and T be a team with the following properties:

1. $\{w_1, \overline{w}_1, w_2, \overline{w}_2, \ldots, w_n, \overline{w}_n\} \subseteq W$,
2. $p_i^1 \in \pi(w_i)$, $p_i^1 \notin \pi(w)$ *for* $1 \leq i \leq n$ *and* $w \in W \setminus \{w_i\}$,
   $p_i^0 \in \pi(\overline{w}_i)$, $p_i^0 \notin \pi(w)$ *for* $1 \leq i \leq n$ *and* $w \in W \setminus \{w_i\}$,
3. $\bigcup_{\substack{1 \leq i,j,k \leq n \\ i \neq k}} \{(w_i, x_j), (w_i, x_k), (\overline{w}_i, x_j), (\overline{w}_i, \overline{x}_k)\} \subseteq R$,
4. $w_i \in T$ *or* $\overline{w}_i \in T$, *but not* $w_i, \overline{w}_i \in T$ *for* $1 \leq i \leq n$.

Then the following holds:

$$\exists \alpha \in \{0,1\}^n : \alpha \models \psi \Leftrightarrow \exists \mathcal{M}\ \exists T \subseteq W : \mathcal{M}, T \models f^{\sim}(\psi)$$

Proof. Let ψ be a *3CNF* formula.
"$\Rightarrow$"

Let α be a satisfying variable assignment for ψ. Then we construct a Kripke model $\mathcal{M}$ as requiered. To satisfy $\mathcal{M}, T \models f^{\sim}(\psi)$ the team T is constructed as follows:

$$T = \begin{cases} T \cup \{w_i\} & \text{, if } \alpha(x_i) = 1 \\ T \cup \{\overline{w}_i\} & \text{, if } \alpha(x_i) = 0 \end{cases}$$

Now we have to show that it holds that if p is satisfied then $f^{\sim}(p)$ is satisfied and for $\overline{p}$ analogous.

- Let $\alpha(x_i) = 1$. Then T and $\mathcal{M}$ have to satisfy $\mathcal{M}, T \models {\sim}\overline{p}_i^1 \wedge \overline{p}_i^0$. By construction it holds that $w_i \in T$ and $p_i^1 \in \pi(w_i)$, which implies that ${\sim}\overline{p}_i^1$ is satisfied. The other part of the formula is satisfied, because it holds that $\overline{w}_i \notin T$ and $\overline{w}_i$ is the only world in $\mathcal{M}$ with $p_i^0 \in \pi(\overline{w}_i)$.
- $\alpha(x_i) = 0$ follows from the same argument as for $\alpha(x_i) = 1$, but the following formula has to be satisfied $\overline{p}_i^1 \wedge {\sim}\overline{p}_i^0$ instead.

"$\Leftarrow$"

Let $\mathcal{M}, T \models f^{\sim}(\psi)$ and $\mathcal{M}$, T satisfy the required properties. A satisfying variable assignment α is constructed as follows:

$$\begin{aligned} \alpha(x_i) = 1 & \quad \text{, iff } w_i \in T, \\ \alpha(x_i) = 0 & \quad \text{, iff } \overline{w}_i \in T. \end{aligned}$$

Now we show that if $\mathcal{M}, T \models f^{\sim}(\psi)$ holds, then $\alpha \models \psi$ holds.

- Let $\mathcal{M}, T \models {\sim}\overline{p}_i^1 \wedge \overline{p}_i^0$. This can only be the case if $w_i \in T$ and $\overline{w}_i \notin T$, because ${\sim}\overline{p}_i^1$ requires w_i to be in T and with $\mathcal{M}, T \models \overline{p}_i^0$ it follows that $\overline{w}_i$ cannot be in T. Then it holds that $\alpha(x_i) = 1$.

- The case $\mathcal{M}, T \models \overline{p}_i^1 \wedge {\sim}\overline{p}_i^0$ is analogous to the other case. $\square$

Definition 4.1.3 ($f^{\Diamond}$ transformation).
For every propositional logic formula φ in negation normal form, the transformation $f^{\Diamond}(\varphi)$ is inductively defined as:

$$\begin{aligned} p &\mapsto \Diamond p^1, \\ \overline{p} &\mapsto \Diamond p^0, \\ \theta_1 \vee \theta_2 &\mapsto f^{\Diamond}(\theta_1) \otimes f^{\Diamond}(\theta_2), \\ \theta_1 \wedge \theta_2 &\mapsto f^{\Diamond}(\theta_1) \wedge f^{\Diamond}(\theta_2). \end{aligned}$$

Lemma 4.1.4.
Let ψ be a 3CNF Boolean formula in negation normal form, and $\mathcal{M} = (W, R, \pi)$ be a Kripke model and T be a team with the following properties:

1. $\{w_1, \overline{w}_1, w_2, \overline{w}_2, \dots, w_n, \overline{w}_n, x_1, \overline{x}_1, x_2, \overline{x}_2, \dots, x_n, \overline{x}_n\} \subseteq W$,
2. $p_i^1 \in \pi(x_i)$, $p_i^1 \notin \pi(x)$ *for* $1 \leq i \leq n$ *and* $w \in W \setminus \{x_i\}$,
   $p_i^0 \in \pi(\overline{x}_i)$, $p_i^0 \notin \pi(w)$ *for* $1 \leq i \leq n$ *and* $w \in W \setminus \{x_i\}$,
3. $x_i \in T$ *or* $\overline{x}_i \in T$, *bot not* $x_i, \overline{x}_i \in T$ *for* $1 \leq i \leq n$.

Then the following holds:

$$\exists \alpha \in \{0,1\}^n : \alpha \models \psi \Leftrightarrow \exists \mathcal{M}\ \exists T \subseteq W : \mathcal{M}, T \models f^{\Diamond}(\psi)$$

Proof. Let ψ be a *3CNF* formula.
"$\Rightarrow$"

Let α be a satisfying variable assignment for ψ, then we construct a Kripke model $\mathcal{M}$ as requiered. To satisfy $\mathcal{M}, T \models f^{\Diamond}(\psi)$ team T is constructed as follows:

$$T = \begin{cases} T \cup \{w_i\} & \text{, if } \alpha(x_i) = 1 \\ T \cup \{\overline{w}_i\} & \text{, if } \alpha(x_i) = 0 \end{cases} \quad , i = 1, \dots, n$$

Now we have to show that it holds that if p is satisfied then $f^{\Diamond}(p)$ is satisfied, for $\overline{p}$ analogously.

- Let $\alpha(x_i) = 1$. Then T and $\mathcal{M}$ have to satisfy $\mathcal{M}, T \models \Diamond p_i^1$. With respect to the variable assignment α it holds that $w_i \in T$ and $\overline{w}_i \notin T$. This implies, by construction of $\mathcal{M}$,that all $w \in T$ are connected with x_i. With $p_i^1 \in \pi(x_i)$ it follows that $\mathcal{M}, T \models \Diamond p_i^1$. It also holds that $\mathcal{M}, T \not\models \Diamond p_i^1$, because not all $w \in W$ are connected to $\overline{x}_i$ ($(w_i, \overline{x}_i) \notin R$).

- $\alpha(x_i) = 0$ follows for the same reason as for $\alpha(x_i) = 1$, but $\Diamond p_i^0$ has to be satisfied $\Diamond p_i^0$ instead.

"$\Leftarrow$"

Let $\mathcal{M}, T \models f^{\Diamond}(\psi)$ and $\mathcal{M}$, T satisfying the required properties. A satisfying variable assignment α is constructed as follows:

$$\begin{aligned} \alpha(x_i) = 1 &\quad \text{, iff } w_i \in T, \\ \alpha(x_i) = 0 &\quad \text{, iff } \overline{w}_i \in T. \end{aligned}$$

Now we show that if $\mathcal{M}, T \models f^{\Diamond}(\psi)$ holds, then $\alpha \models \psi$ holds.

- Let $\mathcal{M}, T \models \Diamond p_i^1$. This implies, that all $w \in T$ are connected to x_i, because x_i is the only world labeled with p_i^1. This can only be the case if $w_i \in T$ ans $\overline{w}_i \notin T$. Then it holds that $\alpha(x_i) = 1$.
- Let $\mathcal{M}, T \models \Diamond p_i^0$. With the same argumentation as from above it follows that $\alpha(x_i) = 0$. □

.

4.2 MTL-MC is in PSPACE

In this section we show that *MTL-MC* is in *PSPACE* by the recursive top down algorithm *mtl-mc* (1). The algorithm *mtl-mc* checks if the formula ψ is satisfied with the team T on the Kripke model $\mathcal{M}$.

Input : Kripke model $\mathcal{M}$, team T and *MTL* formula ψ
Output: true if and only if $\mathcal{M}, T \models \psi$
1 **if** $\varphi = \psi_1 \vee \psi_2$ **then**
2   *existence* **guess** T_1, T_2 with $T_1 \cup T_2 = T$
3   **return** mtl-mc($\mathcal{M}$,T_1,ψ_1) **and** mtl-mc($\mathcal{M}$,T_2,ψ_2)
4 **else if** $\varphi = \psi_1 \wedge \psi_2$ **then**
5   **return** mtl-mc($\mathcal{M}$,T,ψ_1) **and** mtl-mc($\mathcal{M}$,T,ψ_2)
6 **else if** $\varphi = \sim\psi$ **then**
7   **return not** mtl-mc($\mathcal{M}$,T,ψ)
8 **else if** $\varphi = \Box\psi$ **then**
9   **return** mtl-mc($\mathcal{M}$,R[T],ψ))
10 **else if** $\varphi = \Diamond\psi$ **then**
11   *existence* **guess** $T' \in \langle \mathrm{T} \rangle$
12   **return** mtl-mc($\mathcal{M}$,T',ψ)
13 **else if** $\varphi = dep(p_1, \dots p_{n-1}, p_n)$ **then**
14   **for** $1 \leq i \leq |T|$ **do**
15   Save for $w_i \in W : (p_1, \dots p_{n-1}, p_n)$
16   **for** $i \leq j \leq |T|$ **do**
17   **if** $(p'_1, \dots p'_{n-1}) = (p_1, \dots, p'_{n-1})$ **then**
18   **if** $p'_n \neq p_n$ **then return** false
19 **else if** $\varphi = p$ **then**
20   **foreach** $w_i \in T$ **do**
21   **if** $p \notin \pi(w_i)$ **then return** false
22   **return** true
23 **else if** $\varphi = \overline{p}$ **then**
24   **foreach** $w_i \in T$ **do**
25   **if** $p \in \pi(w_i)$ **then return** false
26   **return** true

Algorithm 1: *MTL-MC* is in *PSPACE*

This algorithm runs in polynomial space, which can be seen in Line 9 and 14, because in these lines we allocate more space. In line 9 the construction of the successor team is done in polynomial space, because in the worst case all worlds in $\mathcal{M}$ are in this team, but the Kripke model is part of the input. The same reason can be used in line 14, because we only allocate polynomial space for each world.

4.3 MTL-MC is PSPACE-hard

Theorem 4.3.1 (*MTL-MC*($\{\Diamond, \wedge, \sim\}$) is *PSPACE*-hard).
Proof. Let $\psi = \exists x_1 \forall x_2 \ldots \exists x_n \varphi$ be a 3CNF-QBF instance, where w.l.o.g. n is assumed to be even.

The corresponding *MTL-MC*($\{\Diamond, \wedge, \sim\}$) instance is defined as $(\mathcal{M}, T, \Theta)$ where.

- $\mathcal{M} = (W, R, \pi)$ where

$$
\begin{aligned}
W = & \bigcup_{i=1}^{n} \{d_i^j\}_{1 \leq j \leq n} \cup \\
& \bigcup_{i=1}^{n} \{w_i^j, \overline{w}_i^j\}_{0 \leq j \leq n-i} \cup \\
& \bigcup_{i=1}^{n} \{w_i, \overline{w}_i\} \\
R = & \bigcup_{i=1}^{n} \{(d_i^j, d_i^{j+1})\}_{1 \leq j < i} \cup \\
& \bigcup_{i=1}^{n} \{(d_i^i, w_i^0), (d_i^i, \overline{w}_i^0)\} \cup \\
& \bigcup_{i=1}^{n} \{(w_i^j, w_i^{j+1}), (\overline{w}_i^j, \overline{w}_i^{j+1})\}_{1 \leq j < n-i} \cup \\
& \bigcup_{i=1}^{n} \{(w_i^{n-1}, w_j), (\overline{w}_i^{n-1}, \overline{w}_j)\}_{1 \leq j \leq n} \cup \\
\pi(d_i^j) = & \ \emptyset \ , \text{ for } 1 \leq j \leq i \leq n \\
\pi(w_i) = & \ \pi(w_i^j) = \{p_i^1\} \ , \text{ for } 1 \leq i \leq n \ , \ 0 \leq j \leq n-i \\
\pi(\overline{w}_i) = & \ \pi(\overline{w}_i^j) = \{p_i^0\} \ , \text{ for } 1 \leq i \leq n \ , \ 0 \leq j \leq n-i
\end{aligned}
$$

- $T = \{d_i^0\}_{1 \leq i \leq n}$

- $\Theta = \delta_1$

$$
\begin{aligned}
\delta_i := & \begin{cases} \Diamond((\overline{p}_i^1 \oslash \overline{p}_i^0) \wedge \delta_{i+1} & \text{, iff i is odd} \\ \Delta((\sim\overline{p}_i^1 \wedge \sim\overline{p}_i^0) \oslash \delta_{i+1} & \text{, iff i is even} \end{cases} \\
\delta_n := & f^{\Diamond}(\psi)
\end{aligned}
$$

To illustrate the structure of the generated Kripke model, figure 4.1 shows an example Kripke model generated by the formula $\psi = \exists x_1 \forall x_2 \psi'$, where ψ' is a arbitrary propositional formula.

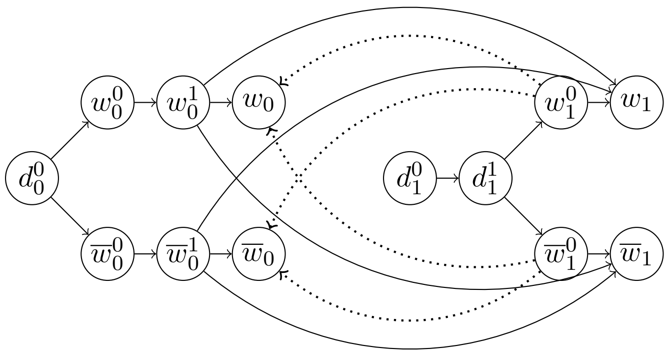


Figure 4.1: Kripke structure for $\exists x_1 \forall x_2$

To proof the correctness we will show, that for any *3CNF-QBF* formula $\psi = \exists x_1 \forall x_2 \ldots \exists x_n \varphi$ it holds that

$$\psi \in \textit{3CNF-QBF} \text{ iff } \mathcal{M}, T \models \Theta$$

"$\Rightarrow$"

Let ψ be an *3CNF-QBF* formula with satisfying assignment tree α. We use this assignment to solve existential quantifications in the formula Θ.

i is odd: $\Diamond(\overline{p}_i^1 \otimes \overline{p}_i^0) \wedge \delta_{i+1}$:

To satisfiy δ_i we have to choose if we extend T by the world $\{w_i\}$ or by $\{\overline{w}_i\}$. If we choose w_i then $\sim\overline{p}_i^0$ is satisfied, $\overline{w}_i$ analogous. We solve this existential quantification problem with respect to the variable assignment α in the following way:

$$T = \begin{cases} T \cup \{w_i\} & \text{,iff } \alpha(x_i) = 1 \\ T \cup \{\overline{w}_i\} & \text{,iff } \alpha(x_i) = 0 \end{cases}$$

i is even: $\Delta(\sim\overline{p}_i^1 \wedge \sim\overline{p}_i^0) \otimes \delta_{i+1})$:

By definition of the universal quantification Δ-operator we have to check that $\delta_\forall$ is satisfied by following teams:

1. $T \cup \{w_i, \overline{w}_i\}$
   We don't want that this case has to satisfy δ_{i+1}, because this isn't a correct variable assignment over x_i. Therefore we catch this case with the sub formula $(\sim\overline{p}_i^1 \wedge \sim\overline{p}_i^0)$. This sub formula is satisfied, because $\sim\overline{p}_i^1$ requires that it holds that $w_i \in T$ and $\sim\overline{p}_i^1$ requires that it holds that $\overline{w}_i \in T$.
2. $T \cup \{w_i\}$ and $T \cup \{\overline{w}_i\}$
   These cases are correct variable assignments over x_i and we wan't that δ_i is only satisfied if δ_{i+1} is satisfied by both teams. This is ensured, because $T \cup \{w_i\}$ and $T \cup \{\overline{w}_i\}$ cannot satisfy $(\sim\overline{p}_i^1 \wedge \sim\overline{p}_i^0)$, because, as mentioned above, a team is required which contains both, w_i and $\overline{w}_i$.

Now by construction of $\mathcal{M}$ and T' we can use Lemma 4.1.4 and it directly follows that:

$$\mathcal{M}, T' \models f^{\Diamond}(\psi)$$

"$\Leftarrow$"

Now we have to show that there exists a variable assignment tree $\alpha \models \psi$ if $\mathcal{M}, T \models \Theta$. $\mathcal{M}, T \models \Theta$ implies that there exists a set of teams where each member T' fulfills $\mathcal{M}, T' \models f^{\Diamond}(\psi)$. If we can show that T' fullfills the properties of Lemma 4.1.4 then we directly get an satisfying variable assigment α. Therefore we show that the cases of the formula Θ generates correct teams T'.

i is odd: $\Diamond(\overline{p}_i^1 \otimes \overline{p}_i^0) \wedge \delta_{i+1}$:

1. $T' \cup \{w_i\}$ or $T' \cup \{\overline{w}_i\}$:
   These cases are correct variable assignments over x_i and they can satisfy δ_i by $\overline{p}^0$ and satisfying δ_{i+1}.
2. $T' \cup \{w_i, \overline{w}_i\}$:
   This case is no correct variable assignment and we have to ensure δ_i cannot be satisfied. Because of $w_i \in T'$ $\overline{p}^1$ cannot by satisfied and $\overline{w}_i \in T'$ prevent $\overline{p}^0$ of satisfying the sub formula.

i is even: $\Delta(\sim\overline{p}_i^1 \wedge \sim\overline{p}_i^0) \otimes \delta_{i+1})$:
The correctness of the universal quantification case is already shown above.

Now it follows directly by Lemma 4.1.4 that $\mathcal{M}, T' \models f^{\Diamond}(\psi)$ holds. □

Example 4.3.2. *Let* $\varphi = \exists x_1 \forall x_2 \exists x_3 (x_1 \vee \overline{x}_3 \vee x_2) \wedge (\overline{x}_1 \vee x_2 \vee \overline{x}_3)$ *be a 3CNF-QBF formula. This formula is satisfiable with* $x_1 = 1$ *and* $x_3 = 0$*. The corresponding Kripke model* $\mathcal{M} = (W, R, \pi)$ *is defined as follows:*

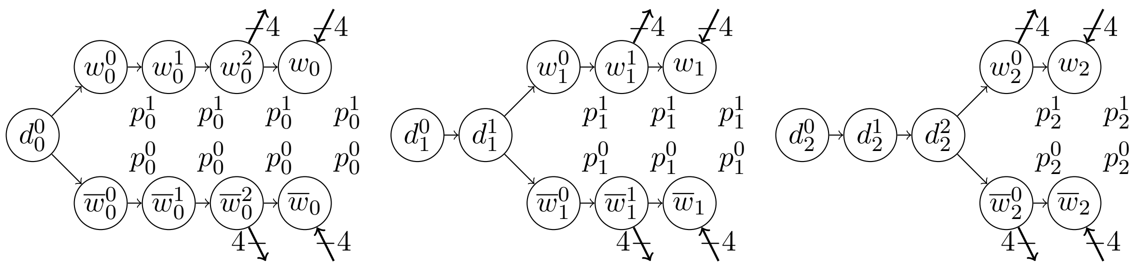


Figure 4.2: Example Kripke model

From the 3CNF-QBF reduction in Lemma 4.3.1 it follows, that the following formula has to be verified on $\mathcal{M}$*:*

$$\begin{aligned}\varphi = {} & \Diamond(\overline{p}_1 \varovee \overline{p}'_1) \wedge \\ & \Delta({\sim}\overline{p}_2 \wedge {\sim}\overline{p}'_2) \varovee \\ & \Diamond(\overline{p}_3 \varovee \overline{p}'_3) \wedge \\ & (f^{\Diamond}(x_1) \vee f^{\Diamond}(\overline{x}_3) \vee f^{\Diamond}(x_2)) \wedge (f^{\Diamond}(\overline{x}_1) \vee f^{\Diamond}(x_2) \vee f^{\Diamond}(\overline{x}_3))\end{aligned}$$

We start with $T = \{w_1, \overline{w}_1, w_2, \overline{w}_2, w_3, \overline{w}_3\}$ as the initial team. To existentially guess x_1 we have to choose that w_1 remains in the team and $\overline{w}_1$ left the team. This is achieved as in proof 4.3.1 with the sub formula "$\Diamond(\overline{p}_1 \varovee \overline{p}'_1) \wedge \delta$". Now we have to universally check the variable x_2. That means, that now two team instances have to be verified. One team which includes w_2 and not $\overline{w}_2$, the other team analogously. We set the last variable x_3, which is an existential guess, to $false$. The development of the start team is shown in Figure 4.3:

$$\{w_1, \overline{w}_1, w_2, \overline{w}_2, w_3, \overline{w}_3\}$$
$$\mid \exists x_1$$
$$\{w_1, w_2, \overline{w}_2, w_3, \overline{w}_3\}$$
$$\forall x_2$$
$$\{w_1, w_2, w_3, \overline{w}_3\} \qquad \{w_1, \overline{w}_2, w_3, \overline{w}_3\}$$
$$\exists x_3$$
$$\{w_1, w_2, \overline{w}_3\} \qquad \{w_1, \overline{w}_2, \overline{w}_3\}$$
$$acc \qquad acc$$

Figure 4.3: Team development

Now we have to check that the model $\mathcal{M}$ and both end teams satisfy following formula:

$$f^{\Diamond}(\psi) = (f^{\Diamond}(x_1) \vee f^{\Diamond}(\overline{x}_3) \vee f^{\Diamond}(x_2)) \wedge (f^{\Diamond}(\overline{x}_1) \vee f^{\Diamond}(x_2) \vee f^{\Diamond}(\overline{x}_3))$$

We have two teams, which have to satisfy the formula.

1. The first team is $\{w_1, w_2, \overline{w}_3\}$. It is easy to see that the variable assignment $\alpha = \{x_1 = 1, x_2 = 2, x_3 = 0\}$ satisfies the formula ψ. Now we have to check that the team also satisfies $f^{\Diamond}(\psi)$. The first clause is satisfied by $f^{\Diamond}(x_1) = \Diamond p_1$, because $w_1 \in T$ and therefore all $w \in T$ are connected to x_i. The second clause is satisfied by $f^{\Diamond}(\overline{x}_3)$ with an analogous argument.

2. Now we have to check the second team $\{w_1, \overline{w}_2, \overline{w}_3\}$, which differs from the first team only in the x_2 position. Now we can use the same satisfying argumentation, because in the first case we didn't use the variable x_2 to satisfy $f^{\Diamond}(\psi)$.

Theorem 4.3.3 (*MTL-MC*($\{\vee, \wedge, \sim\}$) is *PSPACE*-hard).
Proof. Let $\psi = \exists x_1 \forall x_2 \ldots \exists x_n \varphi$ be a *3CNF-QBF* instance, then $\psi \in QBF \Leftrightarrow f(\psi) \in$ *MTL-MC*($\{\vee, \wedge, \sim\}$) The corresponding *MTL-MC*($\{\vee, \wedge, \sim\}$) reduction function is defined as $f(\psi) \mapsto (\mathcal{M}, T, \Theta)$, where

- $\mathcal{M} = (W, R, \pi)$ where

$$
\begin{aligned}
W &= \{x_i, \overline{x}_i\} && 1 \leq i \leq n, \\
R &= \emptyset && \\
p_i &\in \pi(x_i) && 1 \leq i \leq n, \\
p_i^1 &\in \pi(x_i) && 1 \leq i \leq n, \\
p_i^0 &\in \pi(\overline{x}_i) && 1 \leq i \leq n, \\
r_i &\in \pi(x_i), \pi(\overline{x}_i) && 1 \leq i \leq n,
\end{aligned}
$$

- $T = \bigcup_{1 \leq i \leq n} \{x_i, \overline{x}_i\}$,
- $\Theta = \delta_1$

$$
\begin{aligned}
\delta_i :=& \begin{cases} \varphi_\forall \otimes \delta_{i+1} & \text{, iff i is even} \\ \varphi_\exists \vee \delta_{i+1} & \text{, iff i is odd} \end{cases} \\
\delta_n :=& f^{\sim}(\psi)
\end{aligned}
$$

$\varphi_\forall$ has to catch all teams, which contains x and $\overline{x}$ or neither x nor $\overline{x}$.

$$\varphi_\forall := (\sim\overline{p}^1 \wedge \sim\overline{p}^0) \otimes \sim r$$

$\varphi_\exists$ has to separate x and $\overline{x}$ such that the final team does't contain both.

$$\varphi_\exists = p^1 \otimes p^0$$

To proof the correctness we show now, that for any *3CNF-QBF* formula $\psi = \exists x_1 \forall x_2 \ldots \exists x_n \varphi$ it holds that

$$\psi \in \textit{3CNF-QBF} \text{ iff } \mathcal{M}, T \models \Theta$$

"$\Rightarrow$"

Let ψ be a *3CNF-QBF* formula with satisfying assignment tree α. To show that $\mathcal{M}, T \models \delta$ holds, we alternate through the quantifications in $\varphi_\exists$, $\varphi_\forall$ with respect to the variable assignment α.

i is odd: To satisfy $\delta_i = (p^1 \otimes p^0) \vee \delta_{i+1}$ with the team T $(w_i, \overline{w}_i \in T)$, by definition, we have to split T into T_1, T_2 with $T_1 \cup T_2 = T$ and $T_1 \models p_i^1 \otimes p_i^0$ and $T_2 \models \delta_\forall$. With respect to the variable assignment α, we choose $T_1 = \{w_i\}$ if $\alpha(x_i) = 1$ and $T_2 = T \setminus \{w_i\}$. We choose $\overline{w}_i$ analogously if $\alpha(x_i) = 0$.

i is even: To satisfy δ_i, in the even case, with team T, we have to satisfy $T_1 \models (\sim\overline{p}^1 \wedge \sim\overline{p}^0)$ or $T_2 \models \delta_{i+1}$ for all subteams $T_1 \cup T_2 = T$. Now we will show, that only δ_i has to be satisfied if $T_1 = \{w_i\}$ or $T_1 = \{\overline{w}_i\}$, because then T_2 encodes a partially correct variable assignment.

1. Let $w_i, \overline{w}_i \in T_1$ then $\mathcal{M}, T_1 \models ((\sim\overline{p}_i^1 \wedge \sim\overline{p}_i^0)$ holds, because of $w_i \in T_1$ and $\pi(p_i^1) = \{w_i\}$, $\sim\overline{p}_i^1$ is satisfied on T_1. $\sim\overline{p}_i^0$ is also satisfied on T_1, because $\overline{w}_i \in T_1$ and $\pi(p_i^0) = \{\overline{w}_i\}$.
2. Let $w_i \notin T$ and $\overline{w}_i \notin T$, then $\mathcal{M}, T_1 \models \sim r_i$, because $w_i, \overline{w}_i$ are the only worlds labeled with r_i.
3. Let $w_i \in T$ or $\overline{w}_i \in T$ and $|T| > 1$. Then $\mathcal{M}, T_1 \models \sim r_i$, because of $|T| > 1$ there exists, by definition of π, a element $v \in T$ with $v \notin \pi(r_i)$.
4. Let $w_i \in T$ or $\overline{w}_i \in T$ and $|T| = 1$. Then $\mathcal{M}, T_1 \not\models ((\sim\overline{p}_i^1 \wedge \sim\overline{p}_i^0) \otimes \sim r_i)$. In this case we have chosen a correct variable assignment and $\mathcal{M}, T_2$ has to satisfy δ_{i+1}.

Now by Lemma 4.1.2 it holds that $\mathcal{M}, T' \models f^{\sim}(\psi)$, because by construction $\mathcal{M}$ and T fulfill the required properties.

"$\Leftarrow$"

Now we have to show that from $\mathcal{M}, T \models \Theta$ it follows that a variable assignment tree α exists with $\alpha \models \psi$. Now we want to use Lemma 4.1.2 again but therefore we will show that for $\mathcal{M}.T' \models f^{\sim}(\psi)$, our generated team T' fulfills the properties required by the transformation. This is shown by the correctness of the cases of Θ formula.

is is odd: By definition the following has to be satisfied:

$$\mathcal{M}, T_1 \models p_i^1 \otimes p_i^0 \text{ and } M, T_2 \models \delta_{\forall} \text{ with } T = T_1 \cup T_2.$$

There are only two possibilities to satisfy $\mathcal{M}, T_1 \models p_i^1 \otimes p_i^0$. Either $w_i \in T_1, \overline{w}_i \notin T_1$ or $\overline{w}_i \in T_1, w_i \notin T_1$, because w_i is the only world with $p_i^1 \in \pi(w_i)$, $\overline{w}_i$ analogously. This implies that T_2 is always formed like $T \setminus \{w_i\}$ or $T \setminus \{\overline{w}_i\}$. In both cases T_2 fulfills the properties required by the transformation.

i is even: The correctness of the universal quantification case is already shown in the argumentation above.

Now it directly follows from Lemma 4.1.2 that there exists a variable assignment α with $\alpha \models \psi$. □

Example 4.3.4. *Let $\varphi = \exists x_1 \forall x_2 \exists x_3 (x_1 \vee x_2 \vee \overline{x}_3) \wedge (\overline{x}_2 \vee x_3 \vee \overline{x}_1)$ be a 3CNF-QBF formula. The Kripke model $\mathcal{M}$ and the starting team T are as defined in Figure 4.4.*

Satisfying variable assignment for φ *are* $\alpha = \{x_2 = 0, x_3 = 0\}$ *or* $\alpha = \{x_2 = 1, x_3 = 1\}$. *Now we show that the formula* Θ *can be satisfied with these variable assignments.*

$$\begin{aligned}\Theta = {} & (p_1^1 \varovee p_1^0) \vee \\ & ((\sim\overline{p}_2^1 \wedge \sim\overline{p}_2^0) \varovee r_2) \varowedge \\ & (p_3^1 \varovee p_3^0) \vee \\ & (f^{\sim}(\overline{p}_1^1) \varovee f^{\sim}(\overline{p}_2^1) \varovee f^{\sim}(\overline{p}_3^0)) \wedge (f^{\sim}(\overline{p}_2^0) \varovee f^{\sim}(\overline{p}_3^1) \varovee f^{\sim}(\overline{p}_1^0))\end{aligned}$$

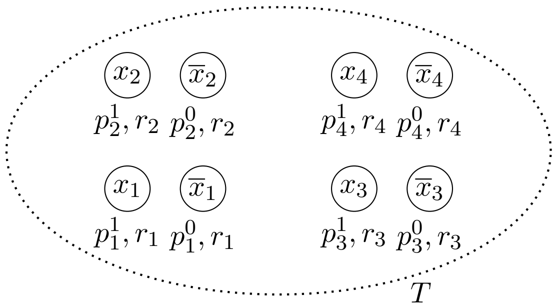


Figure 4.4: Model checking example for operator set $\{\vee, \wedge, \sim\}$

The initial team T is set as $T := \{w_1, \overline{w}_1, w_2, \overline{w}_2, w_3, \overline{w}_3\}$. In the first step of the construction of Θ we have to existentially guess the variable x_1. We choose that x_1 has to be true in the variable assignment and so the team $\{w_1, w_2, \overline{w}_2, w_3, \overline{w}_3\}$ has to satisfy the right part of the split junction. The next step is an universal guess and as shown in the proof above, we now have to check the two teams $T_1 = \{w_1, w_2, w_3, \overline{w}_3\}$ and $T_2 = \{w_1, \overline{w}_2, w_3, \overline{w}_3\}$. The last step is similar to the first and we choose the variable x_3 to be false in the team T_1 and true in the other team. The last thing to check is, that the now constructed teams satisfy the formula $f^{\sim}(\varphi)$. Exemplarily we do this for the variable x_2 with the first team. In our team x_2 is set to false ($\overline{w}_2 \in T_1, w_2 \notin T_1$) and this satisfies the second clause at $f^{\sim}(\overline{p}_2) = \overline{p}_2^1 \wedge \sim\overline{p}_2^0$, because $\overline{p}_2^1$ checks that $w_2 \notin T_1$ and $\sim\overline{p}_2^0$ checks that $\overline{w}_2 \in T_1$.

4.4 Model Checking Operator Fragments

Corollary 4.4.1 (*MTL-MC*($\{\Diamond, \wedge, \sim_k\}$) is Σ_{k+1}^p-hard).
Proof. $QBF_k \leq_m^p$ *MTL-MC*($\{\Diamond, \wedge, \sim_k\}$)
Let $\varphi = \exists x_{11} \ldots \exists x_{1j_1} \forall x_{21} \ldots \forall x_{2j_2} \ldots \Lambda x_{k1} \ldots \Lambda x_{kj_n} \psi,\ \Lambda \in \{\exists, \forall\},\ j_1, \ldots j_n \in \mathbb{N}$ be a QBF_k formula. In this reduction we use nearly the same reduction like in Lemma 4.3.1, where the depth of $\sim$ is not bounded.
We only have to allow that quantification over multiple variables can be expressed in one alternation. Therefore we define the extended model $\mathcal{M} = (W, R, \pi)$ and the reduction formula Θ as:

- $\mathcal{M} = (W, R, \pi)$

$$\begin{aligned}
W = & \bigcup_{i=1}^{n}\bigcup_{l=1}^{j_i}\{d_{i,l}^j\}_{1\le j\le n} \cup \\
& \bigcup_{i=1}^{n}\bigcup_{l=1}^{j_i}\{w_{i,l}^j, \overline{w}_{i,l}^j\}_{0\le j\le n-i} \cup \bigcup_{i=1}^{n}\bigcup_{l=1}^{j_i}\{w_{i,l}, \overline{w}_{i,l}\}, \\
R = & \bigcup_{i=1}^{n}\{(d_{i,l}^j, d_{i,l}^{j+1})\}_{1\le j<i} \cup \\
& \bigcup_{i=1}^{n}\bigcup_{l=1}^{j_i}\{(d_{i,l}^i, w_{i,l}^0), (d_{i,l}^i, \overline{w}_{i,l}^0)\} \cup \\
& \bigcup_{i=1}^{n}\bigcup_{l=1}^{j_i}\{(w_{i,l}^j, w_{i,l}^{j+1}), (\overline{w}_{i,l}^j, \overline{w}_{i,l}^{j+1})\}_{1\le j<n-i} \cup \\
& \bigcup_{i=1}^{n}\bigcup_{l=1}^{j_i}\{(w_{i,l}^{n-1}, w_j), (\overline{w}_{i,l}^{n-1}, \overline{w}_j)\}_{1\le j\le n} \cup, \\
\pi(d_{i,l}^j) = & \emptyset \text{ , for } 1\le j\le i\le n, 1\le l\le j_i \\
\pi(w_{i,l}) = & \pi(w_{i,l}^j) = \{p_{i,l}^1\} \text{ , for } 1\le i\le n \text{ , } 0\le j\le n-i, 1\le l\le j_i \\
\pi(\overline{w}_i) = & \pi(\overline{w}_{i,l}^j) = \{p_{i,l}^0\} \text{ , for } 1\le i\le n \text{ , } 0\le j\le n-i, 1\le l\le j_i t.
\end{aligned}$$

- $T = \{d_{i,l}^0\}_{1\le i\le n}$
- $\Theta = \delta_1$

$$\delta_i := \begin{cases} \Diamond(\bigwedge_{l=1}^{j_i}(\overline{p}_{i,l}^1 \otimes \overline{p}_{i,l}^0) \wedge \delta_{i+1} & \text{, iff i is odd} \\ \Delta(\bigwedge_{l=1}^{j_i}(\sim\overline{p}_{i,l}^1 \wedge \sim\overline{p}_{i,l}^0) \otimes \delta_{i+1} & \text{, iff i is even} \end{cases}$$
$$\delta_n := f^{\Diamond}(\psi)$$

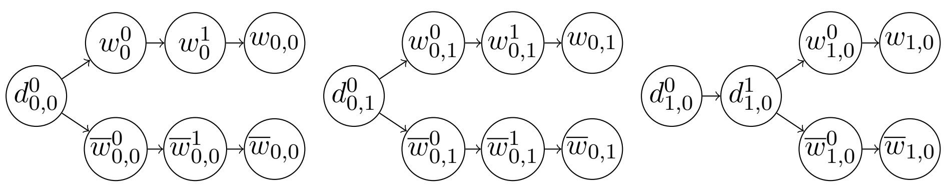


Figure 4.5: Kripke structure for $\exists x_1 x_2 \forall x_3$

The corresponding Kripke model to the formula $\psi = \exists x_1, x_2 \forall x_3 \psi'$ is shown in figure 4.5. This example illustrates how multiple variables are quantified in only one quantification step. For simplicity reasons the connections between the variables and the world labels are not shown.

The proof of the reductions correctness is analogous to the proof in Lemma 4.3.1. It remains to show that the maximum of the reductions negation depth is k.

$$\deg_{\sim}(\Diamond(\bigwedge_{l=1}^{j_i}(\overline{p}_{i,l}^1 \otimes \overline{p}_{i,l}^0) \wedge \delta_{i+1}) = 0$$

$$\deg_{\sim}(\Delta(\bigwedge_{l=1}^{j_i}(\sim\overline{p}_{i,l}^1 \wedge \sim\overline{p}_{i,l}^0) \otimes \delta_{i+1}) = \begin{cases} 1 + \deg_{\sim}(\delta_{i+1}) & \text{, iff } \delta_{i+1} = \delta_n \\ 2 + \deg_{\sim}(\delta_{i+1}) & \text{, otherwise} \end{cases}$$

$$\deg_{\sim}(f^{\Diamond}(\psi)) = 0$$

In the case $\delta_{i+1} = \delta_n$ the negation depth of $\Delta((\sim\overline{p}_k^1 \wedge \sim\overline{p}_k^0) \otimes \delta_{i+1}$ is $1 + \deg_{\sim}(\delta_{i+1})$, because another last quantification change isn't required. $\square$

Example 4.4.2. *In this exmaple we want to check the negation depth of the formula corresponding to $\psi = \exists x_1 \forall x_2 \exists x_3 (x_1 \vee x_2 \vee x_3)$.*

$$\Diamond((\overline{p}_1^1 \otimes \overline{p}_1^0) \wedge \Delta((\sim\overline{p}_2^1 \wedge \sim\overline{p}_2^0) \otimes \Diamond((\overline{p}_3^1 \otimes \overline{p}_3^0) \wedge (\Diamond p_1 \otimes \Diamond p_2 \otimes \Diamond p_3))))$$

As mentioned in the proof from above, the negation depth has to be 2. As the first step we convert, by definition of Δ, $\Delta\psi$ to $\sim\Diamond\sim\psi$

$$\Diamond((\overline{p}_1^1 \otimes \overline{p}_1^0) \wedge \sim\Diamond\sim((\sim\overline{p}_2^1 \wedge \sim\overline{p}_2^0) \otimes \Diamond((\overline{p}_3^1 \otimes \overline{p}_3^0) \wedge (\Diamond p_1 \otimes \Diamond p_2 \otimes \Diamond p_3))))$$

In the last step the second negation is moved deeper insider the formula, to eliminate the one negation.

$$\Diamond((\overline{p}_1^1 \otimes \overline{p}_1^0) \wedge \sim\Diamond((\overline{p}_2^1 \otimes \overline{p}_2^0) \wedge \sim\Diamond((\overline{p}_3^1 \otimes \overline{p}_3^0) \wedge (\Diamond p_1 \otimes \Diamond p_2 \otimes \Diamond p_3))))$$

Now it is easy to see that the negation depth of this formula is 2.

Theorem 4.4.3 (*MTL-MC*($\{\Diamond, \sim, dep\}$) is $P^{NP[1]}$-complete).
Let φ be a MTLformula over $\{\Diamond, \sim, dep\}$. Then the model checking problem MTL-MC($\{\Diamond, \sim, dep\}$) is in $P^{NP[1]}$.

Proof. Let φ be a formula over $\{\Diamond, \sim, dep\}$ and $k_1, \ldots, k_n \in \mathbb{N}$. Then the formula is always either of the form $\varphi = \sim\psi$ or $\varphi = \psi$, where

$$\psi = \Diamond^{k_1} \Delta^{k_2} \ldots \Delta^{k_n} \lambda, \quad \lambda \in \{p, \overline{p}, dep\}, p \in Var.$$

Then it follows from Lemma 3.2.9, that $\Delta\psi'$ can be replaced by $\Box\psi'$.

$$\psi = \Diamond^{k_1} \Box^{k_2} \ldots \Box^{k_n} \lambda, \quad \lambda \in \{p, \overline{p}, dep\}, p \in Var$$

Now we construct a $P^{MDL\text{-}MC(\Diamond,\Box,dep,\overline{p})[1]}$ turing machine M , which solves the problem. At first TM M writes the formula ψ on the oracle tape. Then the turing machine returns the negation of the oracle's answer.

From *MDL-MC*$(\Diamond, \Box, dep, \overline{p})$, which is *NP*-complete ([EL11]), it follows that *MTL-MC*$(\{\Diamond, \sim, dep\})$ is in $P^{NP[1]}$.

Now we have to show that *MTL-MC*$(\{\Diamond, \sim, dep\})$ is $P^{NP[1]}$ hard. Let $A \in P^{NP[1]}$, the corresponding Turing-Machine M_A generally works as shown in figure 4.6.
Now we have to show that $A \leq_m^p$ *MTL-MC*$(\{\Diamond, \sim, dep\})$. In the polynomial many-one reduction, we can simulate the polynomial part of the Machine. Left is the oracle question and four possible acceptance behaviors of M_A shown in figure 4.7.

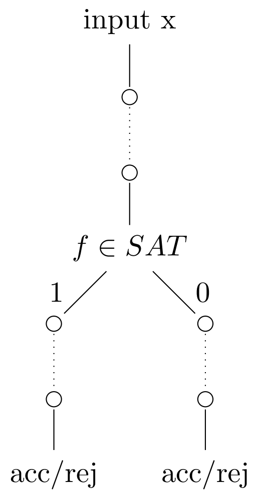
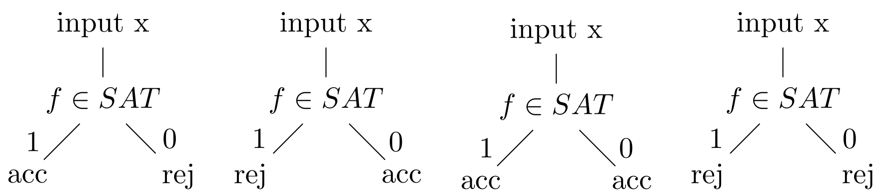


Figure 4.6: $P^{NP[1]}$ machine



(a) $f \in SAT$ (b) $f \in \overline{SAT}$ (c) Accept always (d) Reject always

Figure 4.7: Cases in $P^{NP[1]}$

SAT is represented in *MTL-MC*$(\{\Diamond, \sim, dep\})$ in the same way like in [EL11], but we have to adjust our formula to represent the four acceptance cases.
Let f be the *3CNF* oracle question and $g(f) = \langle \mathcal{M}, T, \varphi \rangle$ the reduction function.
The Kripke structure $\mathcal{M} = (W, R, \pi)$ is defined as follows:

$$W := \{c_1, \ldots, c_n, s_1, \ldots, s_m, \overline{s}_1, \ldots, \overline{s}_m\}$$

$$R \supseteq \begin{cases} \{(c_i, s_j)\} & \text{, if } x_j \text{ occurs in } C_i \\ \{(c_i, \overline{s}_j)\} & \text{, if } \overline{x}_j \text{ occurs in } C_i \end{cases}$$

$$\pi(s_i) \supseteq \{p_i, q\}$$

$$\pi(\overline{s}_i) \supseteq \{p_i\}$$

The initial team is defined by the worlds, which representing the clauses of f.

$$T := \{c_1, \ldots, c_n\}$$

$$\varphi = \begin{cases} \Diamond dep(p_1, \ldots, p_m, q) & , f \in SAT \\ \sim\Diamond dep(p_1, \ldots, p_m, q) & , f \in \overline{SAT} \\ \top & , \text{accept always} \\ \bot & , \text{reject always} \end{cases}$$

The correctness follows directly from the *MDL-MC*($\{\Diamond, \sim, dep\}$) correctness proof in [EL11] and the definition of $\sim$. □

Corollary 4.4.4 (*MTL-MC*($\{\wedge, \sim\}$) is in *P*).
Proof. For this proof we construct an algorithm 2 that runs polynomial time.

Input : Kripke model $\mathcal{M}$, team T and *MTL* formula ψ
Output: true if and only if $\mathcal{M}, T \models \psi$

if $\varphi = \psi_1 \otimes \psi_2$ **then**
  return mtl-mc($\mathcal{M}$,T,ψ_1) or mtl-mc($\mathcal{M}$,T,ψ_2)
else if $\varphi = \psi_1 \wedge \psi_2$ **then**
  return mtl-mc($\mathcal{M}$,T,ψ_1) and mtl-mc($\mathcal{M}$,T,ψ_2)
else if $\varphi = p$ **then**
  for $s \in T$ **do**
    if $p \notin \pi(s)$ **then** return false
  return true
else if $\varphi = \overline{p}$ **then**
  for $s \in T$ **do**
    if $p \in \pi(s)$ **then** return false
  return true
else if $\varphi = \sim p$ **then**
  return not mtl-mc($\mathcal{M}$,T,p)
else if $\varphi = \sim\overline{p}$ **then**
  return not mtl-mc($\mathcal{M}$,T,$\overline{p}$)

Algorithm 2: Model checking algorithm with operators $\wedge$ and $\sim$

It is easy to see that algorithm 2 runs in polynomial time, because we only extend the ordinary propositional logic algorithm with two new cases, which negate the result of the original cases. □

Corollary 4.4.5 (*MTL-MC*($\{\Diamond, \sim\}$) is in *P*).
Proof. Let φ be a formula over $\{\Diamond, \sim\}$ and $k_1, \ldots, k_n \in \mathbb{N}$. Then the formula is always of the form $\varphi = \sim\psi$ or $\varphi = \psi$, where

$$\psi = \Diamond^{k_1}\Box^{k_2}\ldots\Box^{k_n}\lambda, \quad \lambda \in \{p, \overline{p}\}$$

The same argument is used in proof 4.4.3. By [EL11] we can check ψ in polynomial time with *MDL-MC*($\Diamond, \Box, \overline{p}$).
Therefore we construct a oracle turing machine M with an oracle for *MDL-MC*($\Diamond, \Box, \overline{p}$), which runs in polynomial time and solves $\varphi \in$ *MTL-MC*($\{\Diamond, \sim\}$). At first M writes the formula ψ on the oracle tape. Then the turing machine returns the negation of the oracles answer. □

5 Conclusion

We have shown in chapter 3 that *MTL* in general is not closed under the closure properties we looked at. But in Theorem 3.2.9 we have seen, that some parts of *MTL* are downwards closed, there is a possibility for further research. In example the concept of coherence, which Kontinen studied in [Kon10] would be an interesting property.
In chapter 4 we have shown that the *MTL* model checking problem is *PSPACE*-complete, but there is still an open case. It is not known in which complexity class the operator fragment $\{\vee, \sim\}$ is located. Furthermore the complexity could depend of the Kripke model's type. We could restrict the structure such that only $S4, S5, \ldots$ structures are possible.

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