Gottfried Wilhelm Leibniz Universität Hannover
Institut für Theoretische Informatik

# Resource Allocation Problems

Bachelorarbeit

**Sean Pascal Westphal**
Matrikelnr. 3129930

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| Erstprüfer: | Prof. Dr. rer. nat. Heribert Vollmer |
| Zweitprüfer: | PD Dr. rer. nat. habil. Arne Meier |
| Betreuer: | Timon Barlag, M. Sc. |

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# 1 Introduction

At the intersection of economics, philosophy, and computer science sits *social choice theory*, concerned with various problems involving the aggregation of individual preferences to reach some form of collective decision. Within this field exists the *theory of fair allocation*, attempting to answer a question that has been a part of human society since ancient times: How does one fairly divide a finite resource among people, considering their differing needs, rights, and even the fact that they might attempt to cheat the system?

This thesis aims to provide further insight, aimed at readers from the field of computer science without much knowledge of economics theory, into some of the classes of allocation problems briefly listed in Ch. 11 of the *Handbook of Computational Social Choice* [1] by Brandt et al. To this end, after a brief introduction to the *theory of fair allocation* in Ch. 2, some central approaches and their resulting rules will be detailed for each of the chosen classes in Ch. 4. Finally, to build onto the intention of furthering cross-fertilization between computer science and economics voiced in the *Handbook of Computational Social Choice*, Ch. 5 features a brief overview of some recent developments specifically in the computer science branch of social choice, *computational social choice*, with a focus on the classes introduced in this theses or those adjacent to them.

# 2 Theory of Fair Allocation

## 2.1 Introduction to the Theory of Fair Allocation

Before delving into the classes of allocation problems mentioned in the *Handbook of Computational Social Choice* [1], it is necessary to give an overview of what the theory of fair allocation is concerned with in general, and more specifically in which context it will be examined in this thesis.

Generally and informally fair allocation deals with problems concerning the fair division of some form of resource among agents. At first, this might seem very simple: Everyone can receive the same amount. The issues turning this premise into something complex arise from the facts that resources are not always homogeneous or easily divisible and agents can have differing, not easily compatible preferences. Consider for example the division of an inheritance among heirs whose total claims, based on a will, exceed the actual inheritance. Or the common problem of workers applying to firms — no longer can the jobs simply be divided according to the workers' preferences, as the firms themselves may also have preferences for different workers, and those may not line up with the workers' own interests. Both of these Problems will be examined in more detail later.

This also poses the question of what precisely constitutes fairness in a specific real-world context. While this is indeed a complex topic of its own [2], it shall not be of concern here. Instead, mathematical representations of various fairness notions shall be introduced later in the context of axioms, without any value judgement on how important any of these notions are.

Furthermore, the question of whether or not agents are truthful in their indication of their preferences divides the literature on this topic in two branches. One is the strategic branch which focuses on agents' ability to control resources and information and how to prevent the impact of them using that ability to manipulate the distribution in their favor. The other is the normative branch, which is more concerned with the distribution of welfare over a population, irregardless of potential strategic acting by the individual agents [1, 3]. In line with the *Handbook of Computational Social Choice*, this thesis will focus on the normative branch, however strategic issues will not be disregarded entirely when the chosen literature examines them.

## 2.2 Definitions

Additionally, the usual structure through which resource allocation problems will be modeled in this thesis needs to be established. To this end, a generic allocation problem will be described with all its components. The classes that are subject of the next chapter may in some cases impart further restrictions on the model. Unless stated otherwise, all definitions in Ch. 2.2 and Ch. 2.3 will be based on Chapter 11 of the *Handbook of Computational Social Choice* [1].

As fair allocation is an important problem of economics [4] and usually applied to problems with an economic background, the model as a whole is referred to as an *economy*. An economy may have some or all of the following components:

1. *Agents:* An agent refers to an entity — this may be an individual person, a company, a government agency, or any other more abstract entity — whose goal it is to maximize resource gain according to their preferences.

2. *Resource data:* Resource data concerns endowments of goods to be assigned to agents. These goods can be of abstract nature, such as an allocation of labor time, or more concrete, such as money.

3. *Ownership data:* Ownership data is data about resource ownership by agents. Ownership can be individual, i. e. a specific agent owning a resource, semi-collective, i. e. a specific group of agents owning a resource, or collective, i. e. the entire set of agents owning a resource. These modes of ownership may be combined within the same economy and may be contested, in which case disagreement results in incompatible claims.

4. *Preference data:* Preference data indicates natural properties that agents' preferences satisfy in each problem class. An example of this would be the single-peakedness of preferences in the *fair division problems with single-peaked preferences* introduced in Ch. 4.2.

5. *Bounds:* There may be lower or upper bounds imposed on the amount of goods assigned to each individual agent.

A specific allocation problem then contains at least agents and resource data. The precise mathematical definition of what constitutes a problem depends heavily on the specific model used, and will therefore be foregone at this point. It should furthermore be noted that there is no claim to this list being exhaustive of the components of resource allocation problems as a whole, nor of every component being described to its broadest extent, rather it contains only those components needed for the problems discussed in this thesis, described to the extent they are used.

## 2.3 Solutions and Rules

A *solution* associates with each problem in a class a non-empty subset of its set of feasible allocations, therefore it achieves a first elimination of alternatives without specifying the process to be used to determine a final outcome - for example one's first criterion might be that there is no agent that receives nothing at all, so the associated solution would eliminate all those allocations in which any agent receives $\emptyset$. A *rule* then is a *single-valued* solution and hence provides a complete answer on how to allocate resources in each particular problem of a class. Solutions and rules in this thesis will be defined as either the criteria for an allocation to be part of the set resulting from the solution, with $S(P)$ denoting the set of allocations resulting from application of solution $S$ to a given allocation problem $P$, or by describing a process to arrive at the set of allocations.

There are a number of ways of arriving at a solution. For one, one may consider a solution as a formal description of a real-world practice. This can be useful to understand what is done in reality and learn about the desirable as well as the undesirable features of common practices. Useful lessons can be drawn from examining the procedures societies have come up with to resolve resource allocation conflicts, such procedures being for example rules based on equality, proportionality, priority, lotteries, and prices.

There is also the axiomatic approach, that is starting from a list of axioms, where an axiom is a mathematical expression of some notion about how a solution or rule should behave in certain situations. The ultimate goal here is to determine the boundary that separates those combinations of axioms that are compatible from those that are not. These underlying notions vary in scope such that some can be meaningfully expressed in almost any model, while others are a bit more limited. A few broadly scoped notions will be formally introduced in the preliminaries, as they are commonly encountered in literature on various problem classes, and will therefore be relevant in multiple sections of Ch. 3.

Inspiration for solution concepts can also be drawn from cooperative game theory. However, the models studied in that theory are typically abstract in the sense that only sets of achievable utility vectors are given, without specifying a description of the actual physical choices the agents have. Utilizing the theory of cooperative games therefore poses the challenge of finding the most natural expressions, in the context of the model under examination, of the principles that have been important in that theory. Alternatively, the allocation problem under investigation can be mapped into games, where solutions from cooperative game theory can then be applied.

Lastly, rules that are common in one area of resource allocation problems may serve as inspiration for new rules in another. For instance, the theory of two-sided matching has been important in defining solutions for priority-augmented augmentation problems.

# 3 Preliminaries

$N = \{1, \ldots, n\}$ will refer to the set of all agents, and $\Omega$ will refer to the set of goods (also known as a social endowment) to be distributed. An allocation $x$ is then a set of pairs $(i, a), a \in \Omega \cup \emptyset$ assigning all agents $i \in N$ an amount of goods.

A central concept in allocation problems are *preference relations*, an ordering of agents' preferences over $\Omega$, or in the case of two-sided matching problems, over a set of agents (the latter case will be covered in more detail in the preliminaries of Ch. 4.4). The set of preference relations for a given problem will be denoted as $\succsim$, and the preference order of an individual agent $i \in N$ as $\succsim_i$.

$a \succ_i b$ denotes that agent $i$ strictly prefers $a$ over $b$, i. e. values $a$ higher than $b$. $a$ and $b$ can be any assignments or matchings in a given problem, such as amounts of an infinitely divisible resource or, in the case of matching problems, even other agents. Similarly, $a \sim_i b$ will denote that $i$ is indifferent between $a$ and $b$, i. e. values them the same. A weak preference of $a$ over $b$ by agent $i$, i. e. that agent $i$ values $a$ at least as high as $b$, but not necessarily higher, is then denoted as $a \succsim_i b$.

### Fairness axioms

The definitions in this sections are again based on the *Handbook of Computational Social Choice* [1] unless otherwise noted.

For an allocation $x$, let $x_i$ refer to the goods agent $i \in N$ is assigned in $x$.

**Definition 1** (Pareto efficiency)**.** An allocation $x$ is called Pareto efficient for an economy if no agent can be made better off without making another agent worse off, i. e. there exists no alternative allocation $x'$ such that for each $i \in N, x'_i \succsim_i x_i$ while there exists at least one $j \in N, x'_j \succ_j x_j$.

Pareto efficiency is also simply referred to as *efficiency*, with a Pareto efficient allocation also being referred to as *efficient*.

**Definition 2** (Envy-freeness)**.** An allocation $x$ is called envy-free for an economy if no agent values the share of another agent higher than their own, i. e. for all $i, j \in N, x_i \succsim_i x_j$.

**Definition 3** (Strategy-proofness)**.** A solution $s$ is called strategy-proof if agents do not gain anything by being untruthful about their preferences, i. e. for every agent $i \in N$ and their "true" preference relation $\succsim_i$ leading to a set of allocations $X$ under application of $s$, there exists no alternative "fake" preference relation $\succsim_i'$ such that applying the solution with $\succsim_i'$ in replacement of $\succsim_i$ yields a set of allocations $X'$ containing an allocation $x'$ such that $x_i' \succsim_i x_i$[5]

# 4 Classes of Allocation Problems

## 4.1 Classical fair division Problems

### 4.1.1 Overview

The first of the problem classes described in the *Handbook of Computational Social Choice* is that which the Authors call *classical fair division problems*. Due to its simplicity, rather elaborating on it, at this point a simple quote of their given definition seems appropriate:

> A social endowment $\Omega \in \mathbb{R}^{\ell}_{+}$ of $\ell$ infinitely divisible goods has to be distributed among a group $N$ of agents. Each agent $i \in N$ is equipped with a preference relation $\succsim_i$ over an $\ell$-dimensional commodity space $\mathbb{R}^{\ell}_{+}$. These preferences satisfy "classical" assumptions of continuity, monotonicity, and convexity.

Perhaps the most well-known example of a problem satisfying this definition is the *cake-cutting problem*, in which the social endowment $\Omega$ is a cake. The cake does not have to be homogenous, rather it can be thought of as perhaps having unevenly distributed toppings, or an uneven mixing of multiple types of dough. The problem to be solved is then how to fairly divide this cake between multiple agents that might have different preferences about the toppings or dough. This serves as a metaphor for a variety of real-life problems, such as the division of an area of land or broadcast time on a television channel.

As the term *classical* might imply, this class of problems, in particular the *cake-cutting problem* has been well studied in game theory, economics, and computer science over the past decades. Indeed a solution for 2 agents, known as *divide and choose*, which, under certain criteria, is fair, has been known since ancient times, being informally mentioned in the Bible [6] and likely having been known even before then. It will be introduced in some depth via the *cake-cutting problem*, in part to provide a foundation that the following sections can build on.

### 4.1.2 The Model

The model that will be used for the cake-cutting problem will be taken directly from chapter 13 of the *Handbook of Computational Social Choice*[1]. It contains a set of agents $N = \{1, \ldots, n\}$ and the heterogeneously divisible cake represented by the interval $[0, 1]$. Each agent has a valuation function $V_i$ mapping a given subinterval $I \subseteq [0, 1]$ to a value in $[0, 1]$ assigned to it by agent $i$ such that $V_i(\emptyset) = 0$ and $V_i([0, 1]) = 1$. $V_i(x, y)$ will be used as a shorthand for $V_i([x, y])$. These valuation functions are assumed to satisfy normalization, divisibility, nonnegativity and additivity.

**Definition 4** (Normalization)**.** All agents value the entirety of the cake as 1, i. e. for all agents $i$, $V_i(0, 1) = 1$.

**Definition 5** (Nonnegativity)**.** No agent considers any part of the cake to be of negative value, i. e. for every subinterval $I$ and every agent $i$, $V_i(I) \geq 0$.

**Definition 6** (Divisibility)**.** Any part of the cake can be further divided into parts of lesser or equal value, i. e. for every subinterval $[x, y] \subseteq [0, 1]$ and every agent $i$ and all $\lambda$ $0 \leq \lambda \leq 1$ there exists a point $z \in [x, y]$ such that $V_i(x, z) = \lambda V_i(x, y)$.

The property of divisibility implies the valuation functions to be nonatomic, i. e. $V_i(x, x) = 0$ for every $x \in [0, 1]$. This allows disregarding the boundaries of intervals, and in particular treating two intervals as disjoint if their intersection is a singleton.

**Definition 7** (Additivity)**.** Addition of two separate parts of the cake results in the same value for an agent as adding up the values of the parts individually, i. e. for all agents, the value of two disjoint subintervals $I, I'$, $V_i(I) + V_i(I') = V_i(I \cup I')$.

The property of additivity means that for any set of intervals $X$ assigned to an agent, its value is simply $V_i(X) = \sum_{I \in X} V_i(I)$. The set of intervals $X$ will also be referred to as a *piece of cake*.

### Further preliminaries

The fairness axioms of *proportionality* and *equity* will only be relevant in this chapter, and therefore be defined here.

**Definition 8** (Proportionality)**.** An allocation $x$ is called proportional for an economy if every agent receives a share at least proportional in value to the amount of agents, i. e. for all $i \in N$, $V_i(x_i) \geq 1/n$.[1]

**Definition 9** (Equitability)**.** An allocation $x$ is called equitable for an economy if every agent values their own share the same as other agents value theirs, i. e. for all $i, j \in N$, $V_i(x_i) = V_j(x_j)$.

[1] $\sum(V_i(x_i))$ can be greater than 0, as different agents may value different pieces of the cake differently.

### 4.1.3 Classic Cake-Cutting Algorithms

#### Cut and Choose

Cut and choose is a very simple and intuitive algorithm to compute a proportional and envy-free allocation for two agents. Agent 1 cuts the cake into two pieces $X_1$ and $X_2$ such that $V_1(X_1) = V_1(X_2)$. Agent 2 then chooses its preferred piece, and agent 1 receives the remaining piece. Formally, if $V_2(X_1) \geq V_2(X_2)$ then set $A_2 = X_1, A_1 = X_2$, otherwise set $A_1 = X_1, A_2 = X_2$. This allocation is clearly proportional as both agents get a piece they value more or equally to the other out of 2 pieces, and as for two agents the properties of proportionality and envy-freeness are equivalent, it is also envy-free. It is however not equitable as it is possible for agent 2 to value one of the pieces higher than 1/2 the total value, while agent 1 will always get a piece it values at precisely 1/2 the value.

#### Even-Paz

A recursive algorithm proposed by Even and Paz in 1984[7] achieves a proportional allocation for any number of agents. It should be noted here that an algorithm achieving the same, but less computationally efficient[1] has already been defined by Dubins and Spanier in 1961[8], but will not be presented here due to scope constraints.

For ease of exposition, let the number of agents $n$ be a power of 2. The algorithm takes a subset of agents $1, \ldots, k$ and a piece $[y, z]$ and then asks each agent $i$ in the subset to mark a point $x_i$ such that $V_i(y, x_i) = \frac{V_i(y,z)}{2}$. Let $x_{i_1}, \ldots, x_{i_k}$ be the marks sorted such that $x_{i_j} \leq x_{i_j+1}$ for $j = 1, \ldots, k-1$. The algorithm is then recursively called with agents $i_1, \ldots, i_{k/2}$ and the piece $[y, x_{i_{k/2}}]$, and agents $i_{k/2+1}, \ldots, i_k$ and the piece $[x_{i_{k/2+1}}, z]$. When the algorithm is called with a set $i$ only containing a single agent and an interval $I$ it assigns $A_i = I$. The initial call happens with all agents and the entire cake.

At depth $k$ in the recursion tree, $\frac{n}{2^k}$ agents share a piece of cake that each agent values at at least $\frac{1}{2^k}$. At depth $\lg n$ the algorithm is called with one agent and a piece of cake that agent values at at least $\frac{1}{2^{\lg n}} = \frac{1}{n}$. Therefore the Even-Paz algorithm is proportional.

## 4.2 Fair division problems with single-peaked preferences

### 4.2.1 Overview

The second of the problem classes described is that of *fair division problems with single-peaked preferences*. First analysed by Sprumont in 1991[9][10], in contrast to the previously examined classical fair division problems, in fair division problems with single-peaked preferences, the social endowment $\Omega \in \mathbb{R}_+$ consists of only a single, infinitely divisible commodity.[2] Agents' preferences are single-peaked, i. e. they have an ideal amount of the commodity they would like to be assigned. Around this optimum, agents' valuation monotonically decreases. Importantly, it is not possible to discard any of the endowment — the full amount has to be assigned.

The example case given by Sprumont is that of agents contributing to a production process requiring a fixed amount of total work, with each agent having to contribute an assigned amount of labor that they get proportionally compensated for. Sprumont also mentions a slightly different version of the same problem that is found in literature on fixed-price equilibria. This problem is about a two-good exchange economy (hence, $\Omega$ consists of two different commodities) with a rigid relative price.[3] If net demands don't add up to zero, there is a similar need to assign agents unpreferable values for their traded amount. He points out that this problem differs from the previous in two aspects:

1. An agent's net demand may be any real number, as opposed to a value between one and zero representing a percentage of the total work.

2. The net trades must add up to zero, as opposed to the shares having to add up to one.

By imposing the restriction of not rationing any of the agents on the short side of the market, i. e. those agents in possession of the good in short supply, however, the problem once again consists of dividing a fixed amount, in this case of demand or supply, among several agents whose total claims exceed or subceed that amount[4]. If agents' preferences are strictly convex[5], their preferences over the bundles on their budget line[6] are single-peaked.

### 4.2.2 The Model

The Model used in this section will be taken largely from Thomson[13], with some of the symbols adjusted to match the choices in earlier sections. Similarly to the model for classical

[2] Thomson initially defines the social endowment specifically in $\mathbb{R}_{++}$ and later adds that it will be defined over $\mathbb{R}_+$ when it simplifies an issue. Economics literature commonly distinguishes between $\mathbb{R}_{++} = (0, \infty)$ and $\mathbb{R}_+ = [0, \infty)$. For the limited scope in which this thesis examines this class of problems, the distinction is not relevant and as such a definition in $\mathbb{R}_+$ will be assumed for simplicity's sake.

[3] An economy consisting of two goods that can be exchanged for each other, with the exchange rate between the goods not changing.

[4] Whether supply or demand is considered, and subsequently whether the claims exceed or subceed the amount, depends solely on point of reference: Either the consideration is of the long side's demand of the short side's good, which exceeds the short side's supply, or of the short side's supply of the long side's good, which subceeds the long side's demand. Either way, as it has been decided that the agents on the short side will not be rationed, the problem structure is the same.

[5] A preference relation $\succsim$ is called strictly convex if for all $a, b \in \Omega$ such that $a \sim b, a \neq b$, $\lambda a + (1-\lambda)b \succ a$ and $\lambda a + (1-\lambda)b \succ b$ for all $\lambda \in [0, 1]$, i. e. averages are preferred to extremes — if an agent is indifferent between $a$ and $b$, they prefer the weighted average $\lambda a + (1-\lambda)b$ to either $a$ or $b$. [11]

[6] The budget line is a two-dimensional plot over 2 goods, representing the amount of each good that an agent can acquire with their budget [12]. In the given two-good exchange economy, the budget is simply an amount of either of the goods.

fair division problems that was described earlier it contains a social endowment $\Omega \in \mathbb{R}_+$ and a set of agents $N$. Each agent $i \in N$ is equipped with a continuous and single-peaked preference relation $R_i$ defined over the interval $[0, \Omega]$. It follows that there is a number in $[0, \Omega]$ that is the agent's peak amount, denoted $p(R_i)$, such that the agent's valuation strictly monotonically decreases in both directions of that amount, i. e. for each pair $x_i, x'_i \in [0, \Omega]$, if $x'_i < x_i \leq p(R_i)$ or $p(R_i) \leq x_i < x'_i$, then $x_i \succ_i x'_i$.

The class of all such preference relations shall be denoted by $\mathcal{R}$, with an economy being referred to as a list $R \equiv (R_i)_{i \in N} \in \mathcal{R}^N$.

A feasible allocation is a list $x \equiv (x_i)_{i \in N} \in \mathbb{R}_+^N$ such that $\sum x_i = \Omega$, or informally an allocation where the amounts assigned to the agents are not more or less than the available total. The set of feasible allocations will be denoted by $X$.

### 4.2.3 The Uniform Rule

An important rule for division problems with single-peaked preferences is the *uniform allocation rule*. The rule is obtained by specifying a bound that is the same for all agents, either an upper bound if there is too little of the commodity, or a lower bound if there is too much.

**Definition 10** (Uniform rule, $U$)**.** For each $R \in \mathcal{R}^N, x \in U(R)$ if $x \in X$ and there exists a $\lambda \in \mathbb{R}_+$ such that when $\sum p(R_i) \geq \Omega$, then $x = (min\{p(R_i), \lambda\})_{i \in N}$, and when $\sum p(R_i) \leq \Omega$, then $x = (max\{p(R_i), \lambda\})_{i \in N}$

The uniform rule is envy-free, strategy-proof and efficient [14]. It is furthermore both the unique efficient allocation for each economy at which the difference between the greatest and smallest amounts any two agents receive is the smallest, as well as the unique efficient allocation for each economy at which the variance of the amounts received by all the agents is the smallest. [15]

It should be noted that this rule can be criticized based on the fact that it fully satisfies some agents, i. e. gives them their peak amounts — specifically those agents with the lowest peak amounts if $\sum p(R_i) \geq \Omega$ and those with the highest peak amounts if $\sum p(R_i) \leq \Omega$ — while others may be significantly off their peak amounts. Nevertheless, envy-freeness holds, as no agent will ever have an amount that is closer to another agent's peak amount than their own.

## 4.3 Claims Problems

### 4.3.1 Overview

*Claims problems*, the third of the problem classes described, were first analytically examined by O'Neill in 1982[16] based on arbitration problems and solutions presented in the Talmud.[7]

[7] The Talmud is a Jewish religious text that serves as the most important source of Jewish law[17]

As in the previously examined fair division problems with single-peaked preferences, there is a social endowment $\Omega \in \mathbb{R}_+$ consisting of a single, infinitely divisible commodity. Rather than splitting it among a group of agents $N$ with regards to preferences, however, the agents in the group have mutually incompatible claims on it. One of the examples from the Talmud analysed by O'Neill is of a father leaving each of his sons a different amount of his estate, with the total amounts adding up to more than his possessions. Another common problem in this class is the division of assets after a bankruptcy. Intuitively one might think that a simple proportional divide would be the most fair, and the literature does consider this as the *proportional rule*, however it can easily be argued that, by dividing the resource proportionally, the agent with the smallest claim would receive a very small amount in absolute terms, and should at least get a certain minimum compensation. Similarly, one can argue that the absolute damage for the agent with the highest claim is the highest if allocation is proportional, and as such that agent should receive a proportionally larger amount.

This section and the definitions within will be primarily based on a 2008 article by Thomson[18] in which he proposes two families of rules to determine allocations for claims problems, based on the two considerations outlined above. It should be noted here that the two families are not necessarily mutually exclusive, nor are they exhaustive of all possible rules for claims problems, but they serve well as a general overview of and means of contrasting important rules in the literature.

### 4.3.2 The Model

Similar to the two models from the previously discussed classes, the model contains a set of agents $N$. Each agent $i \in N$ has a claim $c_i \in \mathbb{R}_+$, with $c \equiv (c_i)_{i \in N}$ denoting the vector of claims. The social endowment $\Omega$ is insufficient to honor all these claims. A claims problem is then a pair $(c, \Omega)$ such that $\sum c_i \geq \Omega$, with $C$ refering to the set of all claims problems. A rule is a function that associates each problem with a vector $x \in \mathbb{R}^N$ such that $0 \leqq x \leqq c$ and $\sum x_i = \Omega$, with such an $x$ called an awards vector for $(c, \Omega)$. $X$ will denote the set of these vectors.

For sake of simplicity, it will be assumed that agents are ordered such that $c_1 \geq c_2 \geq \cdots \geq c_n$.

### 4.3.3 The ICI family of rules

The first family proposed by Thomson is the *Increasing-Constant-Increasing* family, or ICI for short. It is named such in reference to the behaviour of each agent's assigned share for an increasing $\Omega$, which is divided into three intervals with the first displaying increasing, the second displaying constant, and the third displaying increasing behaviour for the share. The size of any of these intervals may be 0. Informally, starting from $\Omega = 0$, as $\Omega$ increases, at first each agent receives an equal increase in their share. Once a threshold, the value of which depends on the claims problem $(c, \Omega)$, has been reached, the agent with the lowest claim "drops out" and no longer receives an increase. This pattern continues until a threshold is reached for the agent with the highest claim, at which point the other agents "re-enter" starting with the agent with the highest claim, up until $\Omega = \sum(c_i)$ when all agents are fully compensated. A formal definition is available from Thomson [18], inclusion of which would be beyond the scope of this thesis due to its additional prerequisites while not being relevant for anything discussed in this thesis.

An example of a rule in this family would be the *minimal overlap rule* introduced by O'Neill. [16]

**Definition 11** (Minimal overlap rule, MO)**.** Claims on specific parts of the endowment are arranged to minimize overlap between the claims, i. e. either

1. $c_1 \geq \Omega$, in which case claimant 1 claims the interval $[0, \Omega]$ and further claimants $i \in N, i > 1$ claim $[0, min(c_i, \Omega)]$, i. e. claims are nested, or

2. $c_1 < \Omega$, in which case there exists some $t \in [0, \Omega]$ such that each claimant $i \in N$ for which $c_i \geq t$ claims $[0, t]$ as well as a part of $[t, \Omega]$ of size $c_i - t$ such that there is no overlap between these claims, and each claimant $i$ for which $c_i < t$ claims $[0, c_i]$, i. e. each claimant has a claim in $[0, t]$, in which interval claims are nested, and the claimants with the highest total claim also have a claim in $[t, \Omega]$ with no overlap. [19]

To better illustrate this rule, suppose the inheritance problem from Rabbi Abraham Ibn Ezra referred to by O'Neill, in which a man dies and leaves each of his 4 sons different amounts of his estate. One of the sons ($a_1$) has been left all of it, another ($a_2$) has been left half, the third ($a_3$) has been left one third, and the fourth ($a_4$) has been left one quarter. By assigning the total value of the estate as $\Omega = 120$, this results in $c_1 = 120, c_2 = 60, c_3 = 40, c_4 = 30$. Arranging these claims according to the minimal overlap rule yields four parts of the estate to be divided. With parts denoted as $p_i$, the resulting parts are $p_1 = 60, p_2 = 20, p_3 = 10, p_4 = 30$, where each part $p_i$ is claimed by sons $a_1, \ldots, a_i$. Equally dividing each of these parts to each son with a claim on it results in $\frac{p_4}{4} = 7.5$ going to $a_4$, $\frac{p_4}{4} + \frac{p_3}{3} = 10.8\overline{3}$ going to $a_3$, $\frac{p_4}{4} + \frac{p_3}{3} + \frac{p_2}{2} = 20.8\overline{3}$ going to $a_2$, and the rest of $\frac{p_4}{4} + \frac{p_3}{3} + \frac{p_2}{2} + p_1 = 80.8\overline{3}$ going to $a_1$.

An increase in the value of $\Omega$ while keeping the sons' claims fixed results in every son's share going up equally until $\Omega = c_1 + c_4 = 150$, after which point the share of $a_4$ stays constant for further increases to $\Omega$. The same then happens for $a_3$ at $\Omega = c_1 + c_3 = 160$. Once $\Omega = c_1 + c_2 = 180$, $a_1$ receives the entirety of his claim, as there is no longer a need for overlap between $c_1$ and $c_2$. After this point, the share of $a_1$ stops increasing, whereas the shares of $a_2, a_3, a_4$ all increase with increasing $\Omega$, until reaching another threshold $\Omega = c_1 + c_2 + c_4 = 210$ where the share of $a_4$ stops increasing once again. This pattern continues until $\Omega = \sum(c_i) = 250$, at which point every son is able to receive the entirety of his claim.

Another rule in the ICI family, the *Talmud rule*, proposed by Aumann and Maschler in 1985 [20], will also be introduced here as it will be relevant for understanding of a rule in the following section. The definition will again be taken from Thomson [18].

**Definition 12** (Talmud rule, T)**.** Let $T_i(c, \Omega)$ refer to the share assigned to agent $i$ in the allocation given by the Talmud rule for a given claims problem. For each $(c, \Omega) \in C$ and each $i \in N$,

$$T_i(c, \Omega) = \begin{cases} min\{\frac{c_i}{2}, \lambda\} & \text{if } \sum_{i=1}^{n}(\frac{c_i}{2}) \geq \Omega, \\ c_i - min\{\frac{c_i}{2}, \lambda\} & \text{otherwise,} \end{cases}$$

where in each case $\lambda \in \mathbb{R}_+$ is chosen so as to achieve Pareto-efficiency, i. e. such that $\sum_{i=1}^{n}(T_i(c, \Omega)) = \Omega$.

For the previously introduced problem from Ibn Ezra, for an endowment of $\Omega = 120$ this would mean that $a_1$ receives 55, $a_2$ receives 30, $a_3$ receives 20, and $a_4$ receives 15, with $\lambda = 55$. Generally, for increasing $\Omega$ starting from 0, agents "drop out" once their share reaches half their claim, and then "re-enter" once the share of the agent with the highest claim reaches half of both agents' claims combined, i. e. agent $j$ "re-enters" once agent $i$ with the highest claim receives $T_i = \frac{c_i}{2} + \frac{c_j}{2}$.

### 4.3.4 The CIC family of rules

The second family proposed by Thomson is the *Constant-Increasing-Constant* family, or CIC for short. As the name might suggest, the inspiration for this family originated as a reverse of the ICI family, where for increasing $\Omega$ starting from 0, each agent's award is first constant, then increasing, then constant again. Informally, starting from $\Omega = 0$, as $\Omega$ increases, at first only the share of the agent with the highest claim increases. Once a threshold has been reached, the agent with the second highest share "enters" as well and gets an equal increase. This pattern continues until the agent with the lowest claim has also "entered". After this point, once the agent with the lowest claim has been fully compensated, that agent "drops out", which continues until all agents have been fully compensated at $\Omega = \sum(c_i)$. A formal definition would again be out of scope for the same reason, but is available [18].

An example of a rule in this family is the *reverse Talmud rule*. The definition here will be taken from van den Brink et al. [21]

**Definition 13** (Reverse Talmud rule, RT)**.** Let $RT_i$ refer to the share assigned to agent $i$ in the allocation given by the reverse Talmud rule for a given claims problem. For each $(c, \Omega) \in C$ and each $i \in N$,

$$RT_i(c, \Omega) = \begin{cases} max\{0, \frac{c_i}{2} - \lambda\} & \text{if } \sum_{i=1}^{n}(\frac{c_i}{2}) \geq \Omega, \\ c_i - max\{0, \frac{c_i}{2} - \lambda\} & \text{otherwise,} \end{cases}$$

where in each case, $\lambda \in \mathbb{R}_+$ is chosen so as to achieve Pareto-efficiency, i. e. such that $\sum_{i=1}^{n}(RT_i) = \Omega$.

To once again illustrate this rule using Ibn Ezra's problem, for $\Omega = 120$, $a_1$ receives 58.75, $a_2$ receives 28.75, $a_3$ receives 18.75, and $a_4$ receives 13.75, with $\lambda = 1.25$. For $\Omega < 65$, given fixed claims, $a_4$ would receive 0. More generally, for a given agent $j \in N$, $RT_j > 0$ once $\Omega > \sum_{i=1}^{j-1}(\frac{c_i}{2}i)$, i. e. once the share of agent $j-1$ with the next higher claim exceeds $\frac{c_j}{2}$. Once all sons receive a share, their shares increase equally until $\Omega = \sum_{i=1}^{n-1}(\frac{c_i}{2}i) + nc_n = 185$, at which point $a_4$ receives 30 and is therefore fully satisfied, with $a_1$ receiving 75, $a_2$ receiving 45, and $a_3$ receiving 35. The other sons then reach their threshold in order and stop increasing their share until at $\Omega = 250$ all of their claims are fully satisfied.

## 4.4 Matching agents to each other

### 4.4.1 Overview

The last of the problem classes described is that of *matching agents to each other*. The name of this class might make it seem a bit broader than what is meant, as the specific type of problem referred to are so-called *two-sided matching problems*, in which the agents are split into two sets, with the objective of matching agents from one set to agents from the other. Within this class there are a variety of models, differing in aspects such as:

1. Whether it is possible for an agent to not be matched to any agent.
2. Whether matching is one-to-one, such as pairing men and women for a dance class, or one-to-many, such as pairing students to universities. Many-to-many matching is also possible, such as matching workers to firms when workers are able to work for multiple firms at once, while firms can employ more than one worker.
3. Whether preferences are strict or indifference is allowed.
4. Whether some amount of an infinitely divisible good is to be distributed, and if yes, whether the total amount of it is dependent on which pairs are formed by the agents, and whether the agents care only about how much of the good they are assigned or about both the good and their pairing matching.

Adjacent to two-sided matching problems there are also *roommate problems* in which agents from a single set have to be matched to another agent from the same set in a one-to-one pairing.

### 4.4.2 The Model

The Model in this section will be based on Erdil and Ergin [22]. It describes a many-to-one two-sided matching problem that also allows for agents to not be matched at all, with no additional infinitely divisible good to be distributed. Importantly, indifference in preferences is allowed, with the exception of indifference to not being matched at all.

Rather than a single set, the model contains two disjoint finite sets of agents, set $W$ of workers and set $F$ of firms. Let $N = W \cup F$ refer to the set of all agents. Let there be a vector of the number of positions at firms $q = (q_f)_{f \in F}$, with $q_f \geq 1$ denoting the number of positions at firm $f$. There is also a vector $\succsim = (\succsim_n)_{n \in N}$ of weak orders, i. e. complete and transitive relations, where $\succsim_w$ denotes the preference of worker $w$ over $F \cup \emptyset$ and $\succsim_f$ denotes the preference of firm $f$ over $W \cup \emptyset$, called a preference profile. A preference profile will be called strict if $\succsim_n$ is anti-symmetric for each $n \in N$, i. e. there is no longer any indifference. It will be assumed that there is no worker $w$ and firm $f$ such that $w \sim_f \emptyset$ or $f \sim_w \emptyset$. A worker $w$ is said to be acceptable to firm $f$ if $w \succ_f \emptyset$, and similarly a firm $f$ is acceptable to worker $w$ if $f \succ_w \emptyset$. A strict preference profile $\succsim'$ is called a tie-breaking of $\succsim$ if $x \succsim_n y$ implies $x \succsim'_n y$ for all $x, y, n \in N$.

#### Other Preliminaries

**Definition 14** (Stability)**.** Let $\mu(w)$ refer to the firm a worker $w$ is matched to in matching $\mu$, and $\mu(f)$ to the set of workers a firm is matched to in matching $\mu$. A matching $\mu$ is said to be stable if there is no pair $(w, f)$ such that there exists an alternative matching $\mu'$ such that

1. $\mu'(w) \succ_w \mu(w)$
2. $w \succ_f x, x \in \mu(f)$ or, if $f$ has an empty position, $w \succ_f \emptyset$

for any $w \in W, f \in F$.

Informally, there is no pair of a worker and a firm such that the worker prefers the firm over their current matching, and the firm prefers to add the worker to their list of workers in the current matching at the expense of another worker, or over an empty position if the firm has one. [23]

**Definition 15** (Worker optimality)**.** Let $\mu(w)$ refer to the firm a worker $w$ is matched to in matching $\mu$. A matching $\mu$ is said to be Worker optimal ($W$-optimal) if there exists no other matching $\mu'$ such that $\mu'(w) \succsim_w \mu(w)$ for all agents $w \in W$ and $\mu'(w) \succ_w \mu(w)$ for some $w \in W$, i. e. there exists no alternative matching in which no worker would be worse off while at least one worker would be better off. [24]

### 4.4.3 Algorithms for strict and weak preference profiles

#### The Deferred Acceptance Algorithm

To understand the importance of the difference between strict and weak preference profiles, it is important to first introduce an algorithm that is defined only for strict preference profiles. The example here shall be the *deferred acceptance* algorithm proposed by Gale and Shapley in 1962 [25]. It was originally about the problem of colleges accepting students, in order to better match the model used here, colleges and students will be replaced by firms and workers respectively. The algorithm additionally assumes that workers are not permitted to apply to firms that don't consider them acceptable in the first place.

**Definition 16** (Deferred acceptance algorithm, DA)**.** At the first step, all workers $w$ apply to the firm $f$ of their first choice. Each firm $f$ then places the $q_f$ applicants for which its preference is highest on its waiting list, or all applicants if there are fewer applicants than $q_f$, and rejects the rest.

At the $k$th step, each applicant that was rejected at step $k - 1$ applies to their next best acceptable firm. after which each firm places the $q_f$ applicants for which its preference is highest on its waiting list, or all applicants if there are fewer applicants than $q_f$, and rejects the rest.

The algorithm terminates when every applicant is either on a waiting list or has been rejected by every firm they find acceptable.

This algorithm yields a $W$-optimal, and therefore also stable, matching of applicants. It is furthermore strategy-proof from the agents' perspective [26, 27]. The algorithm does not work when indifferences are permitted since behaviour for ties in preferences on either side is undefined. While any weak preference profile can be turned into a strict preference profile through application of a tie-breaking procedure as established in the model, applying the DA algorithm to a tie-breaking is no longer guaranteed to yield a $W$-optimal matching.

The following proof for this is adapted from an introductory example provided by Erdil and Ergyn [22]: Assume that $W$ consists of two workers $i$ and $j$, and $F$ consists of two firms $K$ and $L$. Firm $L$ prefers $i$ over $j$, i. e. $i \succ_L j$, and worker $j$ prefers firm $K$ over firm $L$, i. e. $K \succ_j L$. Firm $L$ is indifferent between $i$ and $j$, i. e. $i \sim_K j$, and worker $i$ is indifferent between $K$ and $L$, i. e. $K \sim_i L$. These relations form the preference profile $\succsim$. If ties are broken alphabetically, i. e. $i \succ'_K j$ and $K \succ'_i L$, to yield the tie-breaking $\succsim'$, applying the DA algorithm to $\succsim'$ assigns worker $i$ to firm $K$ and worker $j$ to firm $L$. This matching achieves $W$-optimality for $\succsim'$, however it is trivial to see that it is not $W$-optimal for $\succsim$, as worker $j$ prefers firm $K$ while being matched to $L$, while worker $i$ is indifferent between the firms. Furthermore, it is not Pareto efficient as welfare can be improved for both $j$ and $L$ without affecting $i$ nor $K$.

| $\succsim_K$ | $\succsim_L$ |
|---|---|
| $i \sim_A j$ | $j$ |
| | $i$ |

| $\succsim_i$ | $\succsim_j$ |
|---|---|
| $A \sim_i K$ | $K$ |
| | $L$ |

| $\succsim'_K$ | $\succsim'_L$ |
|---|---|
| $i$ | $j$ |
| $j$ | $i$ |

| $\succsim'_i$ | $\succsim'_j$ |
|---|---|
| $i$ | $j$ |
| $j$ | $i$ |

Table 4.1: Preference relations for the original preference profile $\succsim$ and the tie-breaking $\succsim'$

| $\succsim$ | $\succsim'$ |
|---|---|
| $(j, K)$ | $(i, K)$ |
| $(i, L)$ | $(j, L)$ |

Table 4.2: The unique $W$-optimal matchings for $\succsim$ and $\succsim'$. The DA algorithm yields the correct unique $W$-optimal matching for $\succsim'$, but as seen here, it differs from the unique $W$-optimal matching for $\succsim$

### The Efficient and Stable Matching Algorithm

To find a stable matching even when preference profiles are not strict, Erdil and Ergyn [22] propose the *efficient and stable matching algorithm* (ESMA). This algorithm employs *Pareto improvement cycles* (PI-cycles) and *Pareto improvement chains* (PI-chains), which will need to be established before the algorithm.

**Definition 17** (Pareto improvement cycle, PI-cycle)**.** Let $\mu$ be a stable matching for some fixed $\succsim$, with $\mu(w)$ referring to the firm that worker $w$ is matched to in $\mu$. A PI-cycle then consists of distinct workers $w_1, \ldots, w_m \equiv w_0 (m \geq 2)$ such that:

1. Each $w_t$ is matched to some firm
2. $\mu(w_t + 1) \succsim_{w_t} \mu(w_t)$ and $w_t \succsim_{\mu(w_{t+1})} w_{t+1}$ for $t \in \{0, 1, \ldots, m-1\}$, i. e. each worker in the cycle weakly prefers the firm matched to the next worker in the chain, and each firm that a worker in the cycle is matched to weakly prefers the previous worker in the cycle to their currently matched one for all workers in the cycle matched to that firm.
3. At least one of $\mu(w_t + 1) \succ_{w_t} \mu(w_t)$ or $w_t \succ_{\mu(w_{t+1})} w_{t+1}$ is true for some $t \in \{0, 1, \ldots, m-1\}$

If there is a PI-cycle, then the matching $\mu$ can be Pareto improved by letting each worker in the cycle move into the firm of the next worker, which yields the improved matching $\mu'$. *Carrying out* a PI-cycle shall refer to to constructing the new matching $\mu'$ in this manner.

**Definition 18** (Pareto improvement chain, PI-chain)**.** Let $\mu$ be a stable matching for some fixed $\succsim$, with $\mu(w)$ referring to the firm that worker $w$ is matched to in $\mu$. A PI-chain then consists of distinct workers $w_1, \ldots, w_m (m \geq 2)$ and a firm $f$ with an empty position such that:

1. $w_1$ is unmatched,
2. $w_t$ is matched with some firm for each $t \in \{2, \ldots, m\}$,

3. $\mu(w_{t+1}) \succsim_{w_t} \mu(w_t)$ and $w_t \succsim_{\mu(w_{t+1})} w_{t+1}$ for $t \in \{1, \ldots, m-1\}$, i. e. each worker in the chain other than $w_m$ weakly prefers the firm matched to the next worker in the chain, and each firm that a worker in the chain is matched to weakly prefers the previous worker in the cycle to their currently matched one for all workers in the cycle matched to that firm,

4. $f \succsim_{w_m} \mu(w_m)$ and $w_m \succsim_f \varnothing$, i. e. worker $w_m$ weakly prefers the firm with an empty position over its current firm and is acceptable to the firm with an empty position.

Similarly to PI-cycles, if there is a PI-chain, then the matching $\mu$ can be Pareto improved by letting each worker other than $w_m$ in the cycle move into the firm of the next worker, and letting $w_m$ move into the firm $f$ with an empty position, to yield the improved matching $\mu'$. Again, *carrying out* a PI-chain refers to constructing the new stable matching $\mu'$ in this manner.

Erdil and Ergin prove that a stable matching is Pareto efficient if and only if it does not admit PI-cycles nor PI-chains, from which they then arrive at the ESMA family of algorithms:

**Definition 19** (Efficient and stable matching algorithm family, ESMA family)**.** First obtain a stable matching by applying the DA algorithm to a tie-breaking. Then find a PI-cycle or a PI-chain in the matching and, if one exists, carry it out to obtain a Pareto improving matching. Repeat this as long as the obtained matching has a PI-cycle or a PI-chain.

It is specifically called an algorithm family here since the ordering of the agents and firms in some steps, such as when finding a PI-cycle or PI-chain, affects the matching yielded by the algorithm. Through specification of a precise selection rule, a specific algorithm in the family is defined.

As the matching yielded by an algorithm in the family has no PI-cycle nor PI-chain, it must be Pareto efficient. Algorithms in the ESMA family are polynomial in the number of workers and total number of positions. In contrast to the DA algorithm, however, they are not strategy-proof, as in the domain of strict preferences, and by extension the generalized domain of weak preferences, no stable matching is strategy-proof when both sides of the market are strategic actors. [27]

### The Worker-Optimal Stable Matching Algorithm

In a similar manner to computing a Pareto efficient matching, it is possible to compute a $W$-optimal and stable matching for a weak preference profile, just like the DA algorithm does for strict preference profiles. Rather than PI-cycles and PI-chains, this utilizes *stable worker improvement cycles* (SWI-cycles) and *stable worker improvement chains*. These operate similarly to their PI counterpart, but require only that the matching yielded from carrying them out is improved from the workers' perspective, disregarding firms' preferences outside of workers needing to be at least acceptable to firms. A formal definition will not be included here due to scope constraints.

Analogously to PI-cycles and PI-chains, Erdil and Ergin prove that a stable matching is $W$-optimal and stable if and only if there are no SWI-cycles nor SWI-chains, from which they then derive the WOSMA family of algorithms:

**Definition 20** (Worker-optimal stable matching algorithm family, WOSMA family)**.** First obtain a stable matching by applying the DA algorithm to a tie-breaking. So long as the stable matching is not $W$-optimal and stable, there will be an SWI-cycle or an SWI-chain. If that is the case, find an SWI-cycle or an SWI-chain and carry it out to obtain a new stable matching that improves the original one from the workers' perspective. Repeat this as long as the obtained stable matching has an SWI-cycle or SWI-chain.

Algorithms in this family yield a $W$-optimal and stable matching as it contains no SWI-cycles nor SWI-chains. Algorithms in the WOSMA family are polynomial in the number of workers and number of firms. Algorithms in the WOSMA family, just like those in the ESMA family, are, however, not strategy-proof even from the workers' perspective as a result of possible ties in firms' preferences. [28] The resulting matching is however not necessarily Pareto efficient, as firms might have been made worse off. It can therefore be improved by applying an ESMA-algorithm to the result, yielding a $W$-optimal and stable Pareto efficient matching.

# 5 Areas of recent research interest

### Fair division of indivisible goods

There has been much ongoing research into the *fair division of indivisible goods*. Algorithms yielding allocations fulfilling various fairness notions have been explored [29], but various open questions remain, particularly regarding the computation of allocations which satisfy Pareto efficiency and so-called *Envy-freeness up to one item*. [30, 31]

### Incremental Stable Matching Problems

Another area of major recent interest are matching problems with regards to changing markets — models, such as the ones introduced in this thesis, are often applicable only to a static market, i. e. one where agents' preferences do not change. The question is then of how to adapt an existing stable matching for an initial market into a new one, while keeping the new stable matching as close as possible to the old one as to minimize the effort involved in transitioning to the new state. These *incremental stable matching problems* have recently been studied with regards to their complexity by a variety of authors. [32, 33, 34]

### Fair division with restricted preferences

The class of fair division problems with single-peaked preferences introduced in this thesis is just one example of a large amount of potential restrictions on preferences. Elkind et al. suggest that there is much potential in examining preference restrictions in more than one dimension, or considering entirely new restrictions originating in graph theory or mathematics. [35]

### Incorporating empirical data

Beyond even the specific field of the theory of fair allocation, pertaining to the entirety of computational social choice, there have been suggestions and efforts made to incorporate more empirical data into computational social choice research in order to consider human behaviour when developing models. To this end, there may be potential in further analysis of the structure of existing data sets. [35, 36]

# 6 Conclusion

A look at four classes of allocation problems shows a large amount of diversity in applications of the theory of fair allocation, even when only looking at a specific model in a selection of classes. Approaches differ in terms of prioritized fairness criteria, and are often are only applicable to very restricted models. As a result they fail to cover the full breadth of real-world problems. Thus, in spite of the age of the basic premise of division of resources among people, there are still many open questions which go beyond the philosophical foundation of what constitutes fairness. Developing algorithms and proving the general feasibility of computing allocations for these problems therefore remains an active area of research with much unrealized potential that also provides an obvious use of theoretical computer science for common problems in society.

# Erklärung der Selbstständigkeit

Hiermit versichere ich, die vorliegende Bachelorarbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel verwendet zu haben. Die Arbeit hat in gleicher oder ähnlicher Form noch keinem anderen Prüfungsamt vorgelegen.

Hannover, den February 10, 2023

Sean Pascal Westphal

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