Characterization of Ehrenfeucht-Fraïssé equivalence for two classes of finite graphs

Malika More

more@laic.u-clermont1.fr - LAIC Université Clermont 1

Joint work with Annie Chateau

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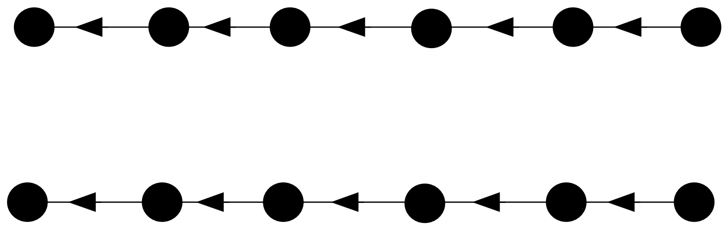
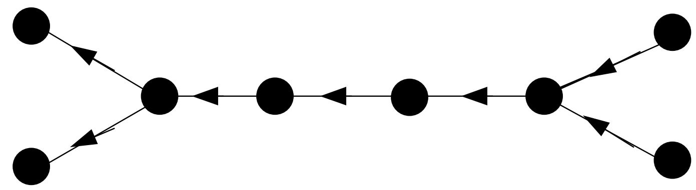
Outline

1 - Introduction

1.1 - k-equivalence

$\sigma = \{E^{(2)}\}$ language of graphs





$\exists x \; (\forall y \; \neg E(x,y) \wedge \exists y \; E(y,x) \;)$

there is a vertex which is not followed by any vertex, but which follows some vertex

$\exists x \; \exists y \; \exists z \; (\; E(x,y) \wedge E(x,z) \wedge y \neq z \;)$

there is a vertex which is followed by two different vertices

quantifier depth

number of nested quantifications

$\exists x \ (\forall y \ \neg E(x,y) \wedge \exists y \ E(y,x) \ )$ $\rightsquigarrow$ qd 2

$\exists x \ \exists y \ \exists z \ (\ E(x,y) \wedge E(x,z) \wedge y \neq z \ )$ $\rightsquigarrow$ qd 3

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k-equivalence

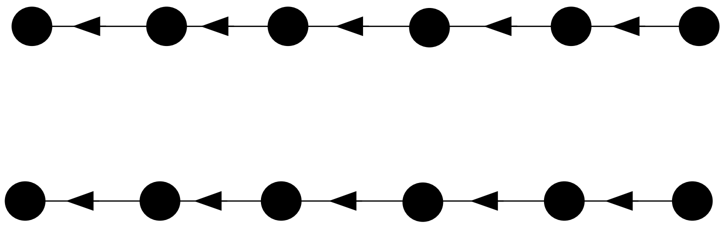
(Fraïssé 54 - Ehrenfeucht 61)

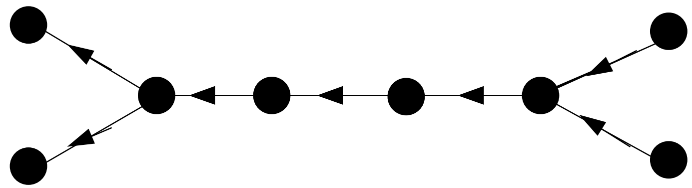
two σ-structures $\mathcal{A}$ and $\mathcal{B}$ are *k-equivalent*

iff

they satisfy the same properties

expressible by a quantifier depth k first-order σ-sentence





$\exists x \; \exists y \; \exists z \; (\; E(x,y) \wedge E(x,z) \wedge y \neq z \;)$ $\quad \rightsquigarrow$ not 3-equivalent

$\exists x \; (\; \forall y \; \neg E(x,y) \wedge \exists y \; E(y,x) \;)$ $\quad \rightsquigarrow$ 2-equivalent ?

Ehrenfeucht Game

Spoiler and Duplicator play k rounds on σ-structures $\mathcal{A}$ and $\mathcal{B}$

At round i :

Spoiler picks up one element in one of the structures

Duplicator picks up an element in the other structure

After k rounds : $(a_1, \ldots, a_k) \in \mathcal{A}$ and $(b_1, \ldots, b_k) \in \mathcal{B}$ in this order

Duplicator *wins* iff $\mathcal{A}/(a_1, \ldots, a_k) \cong \mathcal{B}/(b_1, \ldots, b_k)$

Duplicator has a *winning strategy*

iff Duplicator can win, no matter how Spoiler plays

Theorem

$\mathcal{A}$ and $\mathcal{B}$ are k-equivalent

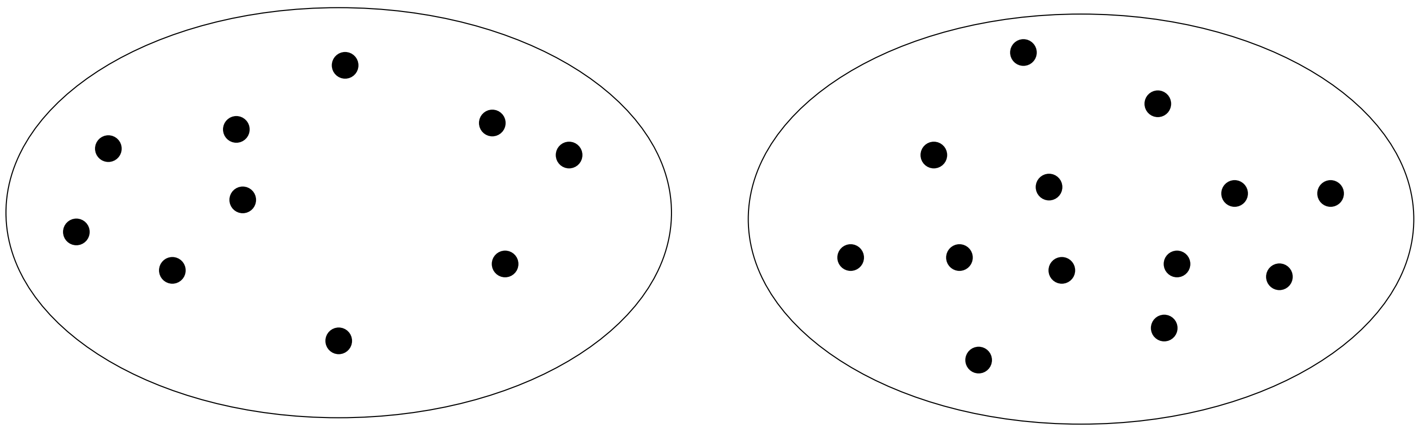
if and only if

Duplicator has a winning strategy.

2-equivalent

1.2 - k-equivalence in classes of finite graphs

Sets



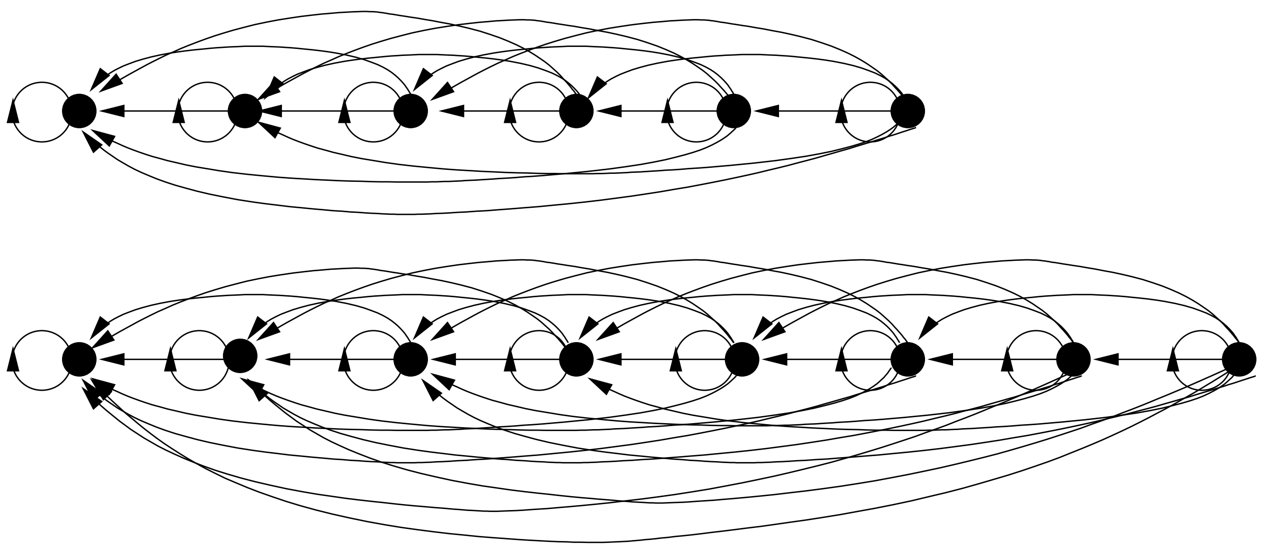
$\mathcal{A}$ and $\mathcal{B}$ finite sets with respective size n and m

$\mathcal{A}$ and $\mathcal{B}$ are k-equivalent

if and only if

$n \neq m \Longrightarrow (n \geq k$ and $m \geq k)$

Linear orderings



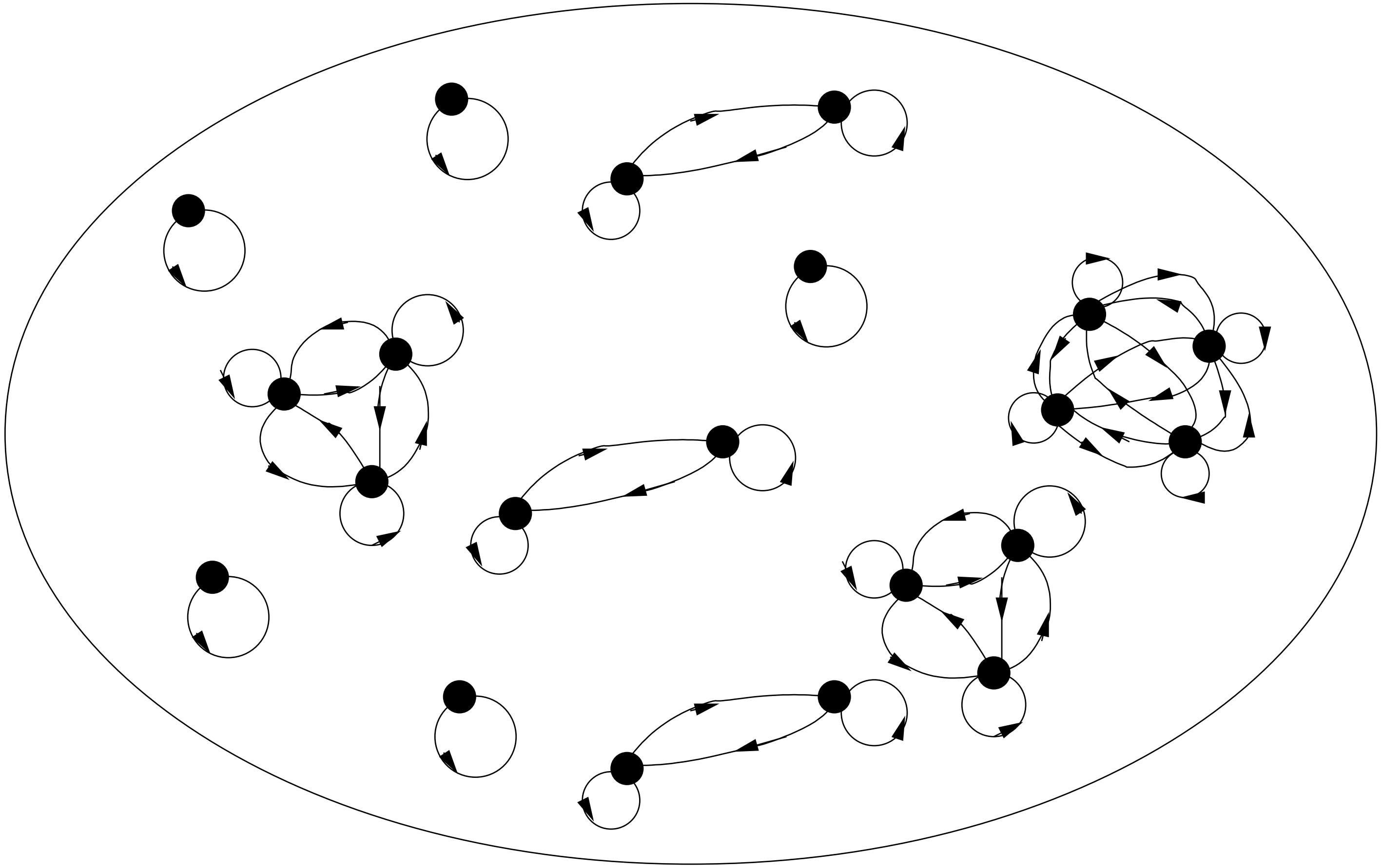
$\mathcal{A}$ and $\mathcal{B}$ finite linear orderings with respective size n and m

$\mathcal{A}$ and $\mathcal{B}$ are k-equivalent

if and only if

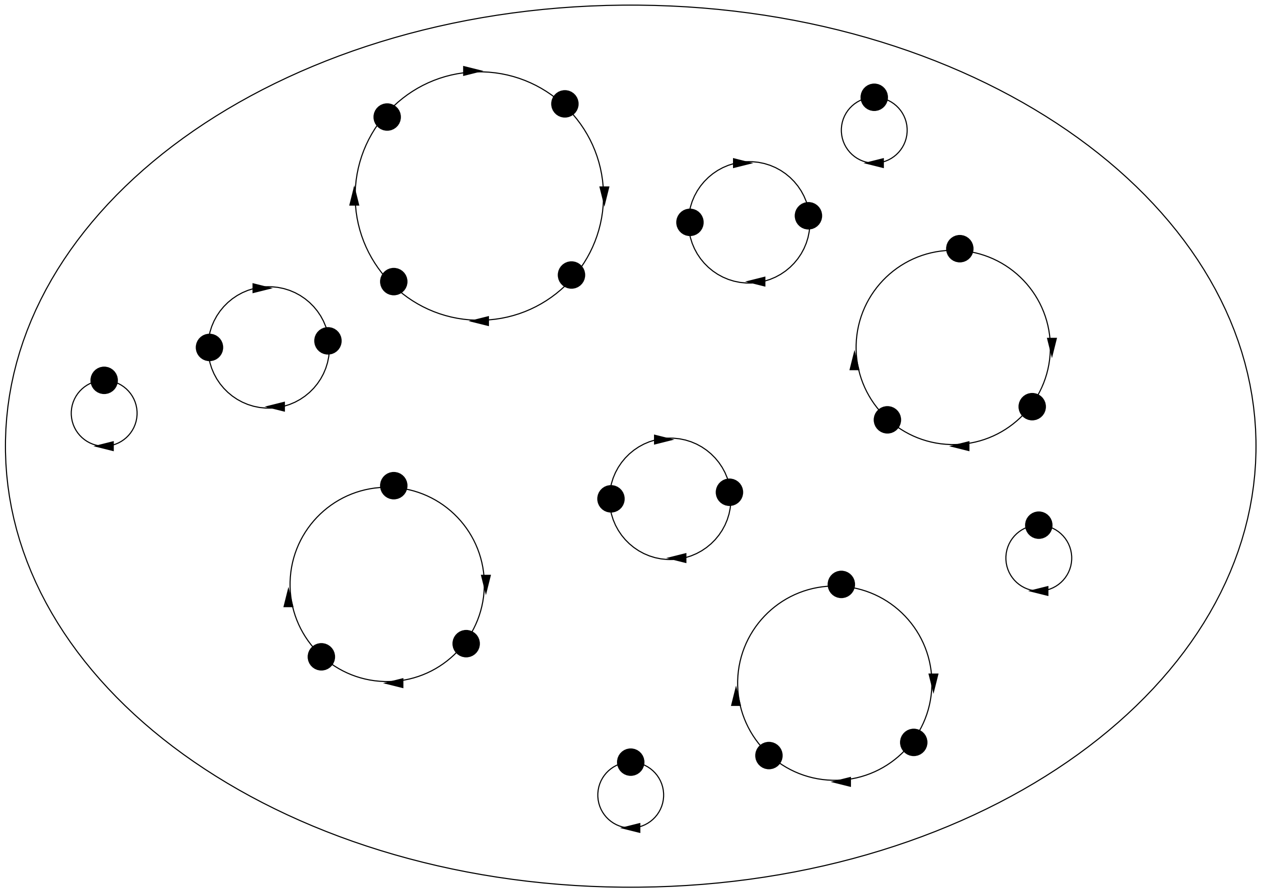
$$n \neq m \Longrightarrow \left(n \geq 2^k - 1 \text{ and } m \geq 2^k - 1\right)$$

Equivalence relation



reflexive - symmetric - transitive

Bijection



indegree 1 and outdegree 1

1.3 - Hanf's locality

Gaifman graph

Gaifman graph of $\mathcal{A} = \langle A, \sigma \rangle$

(x, y) is an edge :

$x = y$ or x, y both in some tuple of some predicate of σ.

r-neigborhood of $x \in A$:

substructure $\mathcal{N}_r^{\mathcal{A}}(x)$ of $\mathcal{A}$ induced by elements

at distance at most r of x in Gaifman graph.

τ *isomorphism type* of r-neighborhood

$n_\tau^{\mathcal{A}}$ number of $x \in A$ s.t. $\mathcal{N}_r^{\mathcal{A}}(x)$ has isomorphism type τ

$(0 \leq n_\tau^{\mathcal{A}} \leq |A|)$

Hanf's locality condition

$c =$ maximal possible size of a 3^k-neighborhood in $\mathcal{A}$ and $\mathcal{B}$.

$\mathcal{A}$ and $\mathcal{B}$ are *Hanf equivalent at rank* k :

for all isomorphism type of 3^k-neighbourhood τ

$$n_\tau^{\mathcal{A}} \neq n_\tau^{\mathcal{B}} \Longrightarrow \left(n_\tau^{\mathcal{A}} \geq c \cdot k \text{ and } n_\tau^{\mathcal{B}} \geq c \cdot k\right).$$

Theorem

$\mathcal{A}$ and $\mathcal{B}$ Hanf equivalent at rank k

$$\Downarrow$$

$\mathcal{A}$ and $\mathcal{B}$ are k-equivalent.

Equivalence relations

Finite equivalence relations $\mathcal{A}$ and $\mathcal{B}$

$$i = 1, \ldots, |\mathcal{A}|$$

$$n_i^{\mathcal{A}} = \text{number of vertices in cliques with a size } i$$

$$\mathcal{A} \text{ and } \mathcal{B} \text{ are } k\text{-equivalent}$$

$$\Uparrow$$

$$\text{for } i = 1, \ldots, \max(|\mathcal{A}|, |\mathcal{B}|)$$

$$n_i^{\mathcal{A}} = n_i^{\mathcal{B}}$$

$$\Updownarrow$$

$$\mathcal{A} \cong \mathcal{B}$$

Bijections

$\mathcal{A}$ and $\mathcal{B}$ finite bijections

$n_i^{\mathcal{A}}$ = number of vertices in cycles with a length i $(1 \leq i \leq 2 \cdot 3^k + 1)$

$n_{2\cdot 3^k+2}^{\mathcal{A}}$ total number of vertices in cycles with a length $\geq 2 \cdot 3^k + 2$

$\mathcal{A}$ and $\mathcal{B}$ are k-equivalent

$$\Uparrow$$

(i) for all $i \leq 2 \cdot 3^k + 1$

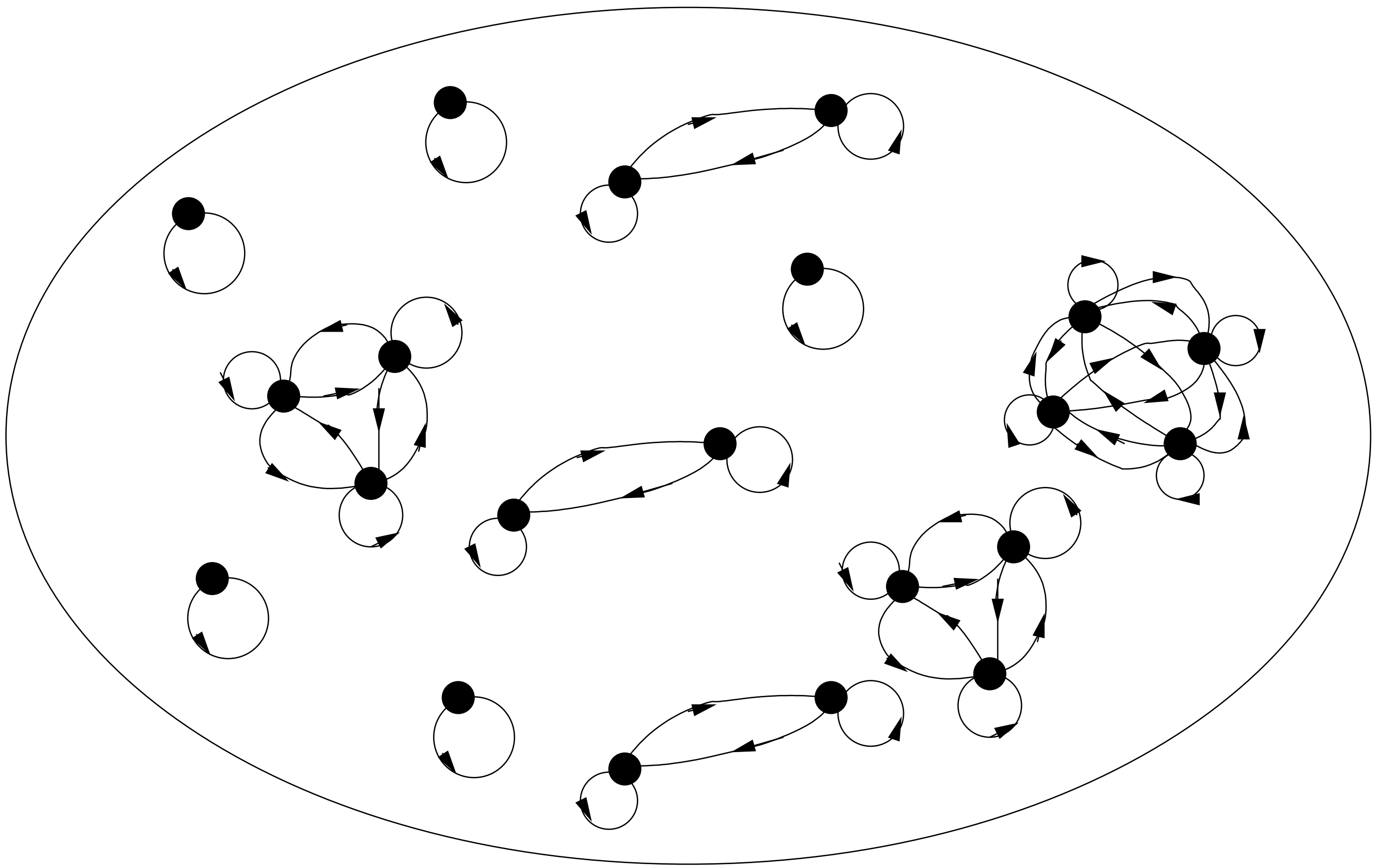
$n_i^{\mathcal{A}} \neq n_i^{\mathcal{B}} \Longrightarrow \left(n_i^{\mathcal{A}} \geq k \cdot (2 \cdot 3^k + 1) \text{ and } n_i^{\mathcal{B}} \geq k \cdot (2 \cdot 3^k + 1)\right)$

(ii) $n_{2\cdot 3^k+2}^{\mathcal{A}} \neq n_{2\cdot 3^k+2}^{\mathcal{B}} \Longrightarrow$

$\left(n_{2\cdot 3^k+2}^{\mathcal{A}} \geq k \cdot (2 \cdot 3^k + 1) \text{ and } n_{2\cdot 3^k+2}^{\mathcal{B}} \geq k \cdot (2 \cdot 3^k + 1)\right)$

2 - Equivalence relations

Equivalence relation



reflexive - symmetric - transitive

Large cliques

with a size $\geq k$

are *unbounded* w.r.t. k-equivalence

Small cliques

with a size $\leq k - 1$

are *definable* by a quantifier depth $k - 1$ formula

$\underline{k\text{-configuration}} : (n_1, n_2, \ldots, n_{k-1}, n_k)$

where

$$\begin{cases} n_i = \text{ number of cliques with a size } i \\ \qquad\qquad \text{for } i = 1 \ldots k-1 \\ n_k = \text{ \textbf{total} number of cliques with a size } \geq k \end{cases}$$

Equivalent k-configurations

1. For all $i \in \{1, \ldots, k-1\}$, we have

$$n_i \neq m_i \Longrightarrow (n_i \geq k - i \text{ and } m_i \geq k - i)$$

2. For all $i \in \{1, \ldots, k\}$, we have

$$\sum_{j=i}^{k} n_j \neq \sum_{j=i}^{k} m_j \Longrightarrow \left(\sum_{j=i}^{k} n_j \geq k - i + 1 \text{ and } \sum_{j=i}^{k} m_j \geq k - i + 1 \right)$$

Theorem

$\mathcal{A}$ and $\mathcal{A}'$ k-equivalent

$$\Updownarrow$$

$C_{\mathcal{A}}$ and $C_{\mathcal{A}'}$ equivalent

Necessary conditions

$Size_i(x) \equiv x$ lies in a clique with a size exactly i $\rightsquigarrow$ qd i

- For some $i \in \{1, \ldots, k-1\}$, assume $n_i \neq m_i$ and $n_i < k - i$

$$\exists x_1 \ldots \exists x_{n_i} \left(\bigwedge_{j=1}^{n_i} Size_i(x_j) \wedge \bigwedge_{j' \neq j} \neg E(x_{j'}, x_j) \wedge \forall z \left(\left(\bigwedge_{j=1}^{n_i} \neg E(z, x_j) \right) \longrightarrow \neg Size_i(z) \right) \right)$$

there are exactly n_i cliques with a size i $\rightsquigarrow$ qd $n_i + i + 1$

- For some $i \in \{1, \ldots, k\}$, assume $\sum_{j=i}^{k} n_j \neq \sum_{j=i}^{k} m_j$ and $\sum_{j=i}^{k} n_j < k - i + 1$

$$\exists x_1 \ldots \exists x_q \left(\bigwedge_{j=1}^{q} \bigwedge_{l=1}^{i-1} \neg Size_l(x_j) \wedge \bigwedge_{j' \neq j} \neg E(x_{j'}, x_j) \right)$$

there are at least q cliques with at least i elements $\rightsquigarrow$ qd $q + i - 1$

Independent conditions

$k = 4$

Configurations $(3,0,0,1)$ and $(4,0,0,2)$:

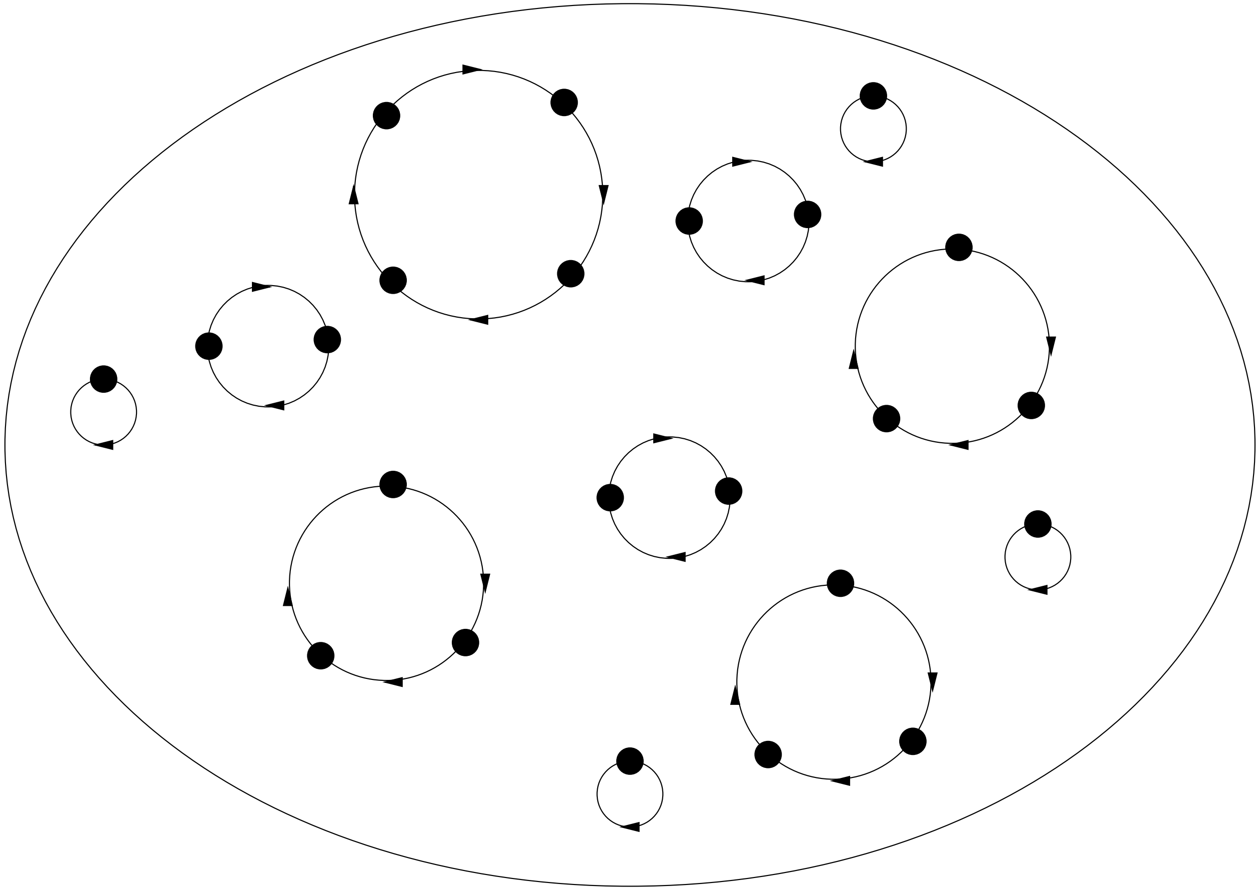
Condition 1 is satisfied but not Condition 2 (for $i = 2$ and $i = 3$)

Configurations $(1,0,0,3)$ and $(0,0,0,4)$:

Condition 2 is satisfied but not Condition 1 (for $i = 1$)

3 - Bijections

Bijection



indegree 1 and outdegree 1

Long cycles

with a length $\geq 2^{k-1} + 3$

are *unbounded* w.r.t. k-equivalence

Short cycles

with a length $\leq 2^{k-1}$

are *definable* by a quantifier depth $k-1$ formula

Medium cycles

with a length $2^{k-1}+1$ or $2^{k-1}+2$

behave

usually like long cycles

in some particular case like short cycles.

<u>k-configuration</u> : $(n_1, n_2, \ldots, n_{2^{k-1}}, n_{2^{k-1}+1}, n_{2^{k-1}+2}, n_{2^{k-1}+3})$

where

$$\begin{cases} n_i = \text{ number of cycles with a length } i \\ \qquad\qquad \text{for } i = 1 \ldots 2^{k-1} + 2 \\ n_{2^{k-1}+3} = \ \textbf{total} \text{ number of cycles} \\ \qquad\qquad \text{with a length } \geq 2^{k-1} + 3 \end{cases}$$

For all $i \leq k-1$ and $l \in \{2^{i-1}+1, \ldots, 2^i\}$

$C_l(x) \equiv x$ lies in a cycle with a length exactly l $\rightsquigarrow$ qd i

$Dist(x, y) \equiv x$ the distance between x and y is exactly l $\rightsquigarrow$ qd $i-1$

	0	$\ldots$	i			$\ldots$	$k-1$			medium	cycles	long cycles
length	1		$2^{i-1}+1$	$\ldots$	2^i		$2^{k-2}+1$	$\ldots$	2^{k-1}	$2^{k-1}+1$	$2^{k-1}+2$	$\geq 2^{k-1}+3$
number	n_1		$n_{2^{i-1}+1}$	$\ldots$	n_{2^i}		$n_{2^{k-2}+1}$	$\ldots$	$n_{2^{k-1}}$	$n_{2^{k-1}+1}$	$n_{2^{k-1}+2}$	$n_{2^{k-1}+3}$

Equivalent k-configurations

1. **(short cycles one by one)** For all $i \leq k-1$ and $l \in \{2^{i-1}+1, \dots, 2^i\}$, we have
   $n_l \neq m_l \Longrightarrow (n_l \geq k-i$ and $m_l \geq k-i)$.

	0	$\dots$	i					$\dots$	$k-1$		
length	1		$2^{i-1}+1$	$\dots$	l	$\dots$	2^i		$2^{k-2}+1$	$\dots$	2^{k-1}
number	n_1		$n_{2^{i-1}+1}$	$\dots$	n_l	$\dots$	n_{2^i}		$n_{2^{k-2}+1}$	$\dots$	$n_{2^{k-1}}$

the number of cycles with a length l is exactly $k-i-1$

2. **(short cycles in bracket)** For all $i \in \{1, \ldots, k-1\}$, we have

$$\sum_{j=2^{i-1}+1}^{2^i} n_j \neq \sum_{j=2^{i-1}+1}^{2^i} m_j \Longrightarrow$$
$$\left(\sum_{j=2^{i-1}+1}^{2^{k-1}+3} n_j \geq k-i+1 \text{ and } \sum_{j=2^{i-1}+1}^{2^{k-1}+3} m_j \geq k-i+1 \right).$$

	i			$\ldots$	$k-1$					
length	$2^{i-1}+1$	$\ldots$	2^i		$2^{k-2}+1$	$\ldots$	2^{k-1}	$2^{k-1}+1$	$2^{k-1}+2$	$2^{k-1}+3$
number	$n_{2^{i-1}+1}$	$\ldots$	n_{2^i}		$n_{2^{k-2}+1}$	$\ldots$	$n_{2^{k-1}}$	$n_{2^{k-1}+1}$	$n_{2^{k-1}+2}$	$n_{2^{k-1}+3}$

the total number of cycles with a length at least $2^{i-1}+1$ is at least $k-i+1$

3. **(isolated largest short cycle)** For all $i \in \{2, \ldots, k-1\}$ and for $l = 2^{i-1} + 1$, we have
if $\left(n_l = \sum_{j=2^{i-2}+1}^{2^{k-1}+3} n_j \text{ or } m_l = \sum_{j=2^{i-2}+1}^{2^{k-1}+3} m_j \right)$, then
$n_l \neq m_l \implies (n_l \geq k - i + 2 \text{ and } m_l \geq k - i + 2)$.

	$i-1$			i					$k-1$					
length	$2^{i-2}+1$	$\ldots$	2^{i-1}	$2^{i-1}+1$	$2^{i-2}+2$	$\ldots$	2^i	$\ldots$						
number	0	$\ldots$	0	$n_{2^{i-1}+1}$	0	$\ldots$	0	$\ldots$	0	$\ldots$	0	0	0	0

the total number of cycles with a length at least $2^{i-2} + 1$ is at least $k - i + 2$

4. **(isolated medium cycle)** For $l = 2^{k-1} + \alpha$ with $\alpha \in \{1, 2\}$, we have

if $\left(n_l = \sum_{j=2^{k-\alpha-1}+1}^{2^{k-1}+3} n_j \text{ or } m_l = \sum_{j=2^{k-\alpha-1}+1}^{2^{k-1}+3} m_j \right)$, then

$n_l \neq m_l \Longrightarrow (n_l \geq 2 \text{ and } m_l \geq 2)$.

	$k-2$			$k-1$					
length	$2^{k-3}+1$	...	2^{k-2}	$2^{k-2}+1$	...	2^{k-1}	$2^{k-1}+1$	$2^{k-1}+2$	$2^{k-1}+3$
$\alpha = 1$	$n_{2^{k-3}+1}$	...	$n_{2^{k-2}}$	0	...	0	$n_{2^{k-1}+1}$	0	0
$\alpha = 2$	0	...	0	0	...	0	0	$n_{2^{k-1}+2}$	0

there is exactly one cycle with a length $2^{k-1} + \alpha$ and no other cycle with a length at least $2^{k-1-\alpha} + 1$

5. **(medium and long cycles in bracket)** We have

$$\sum_{j=2^{k-1}+1}^{2^{k-1}+3} n_j \neq \sum_{j=2^{k-1}+1}^{2^{k-1}+3} m_j \Longrightarrow$$

$$\left(\sum_{j=2^{k-1}+1}^{2^{k-1}+3} n_j \geq 1 \text{ and } \sum_{j=2^{k-1}+1}^{2^{k-1}+3} m_j \geq 1 \right).$$

	medium	cycles	long cycles
length	$2^{k-1}+1$	$2^{k-1}+2$	$\geq 2^{k-1}+3$
number	$n_{2^{k-1}+1}$	$n_{2^{k-1}+2}$	$n_{2^{k-1}+3}$

there is no cycle with a length at least $2^{k-1}+1$

Theorem

$\mathcal{A}$ and $\mathcal{A}'$ k-equivalent

$$\Updownarrow$$

$C_{\mathcal{A}}$ and $C_{\mathcal{A}'}$ equivalent

4 - Ash's counting functions

4.1 - Presentation

Ash's counting function

σ relational signature $\qquad$ k quantifier depth $\qquad$ n finite size

$N_{\sigma,k}(n)$ = number of non k-equivalent σ-structures of size n

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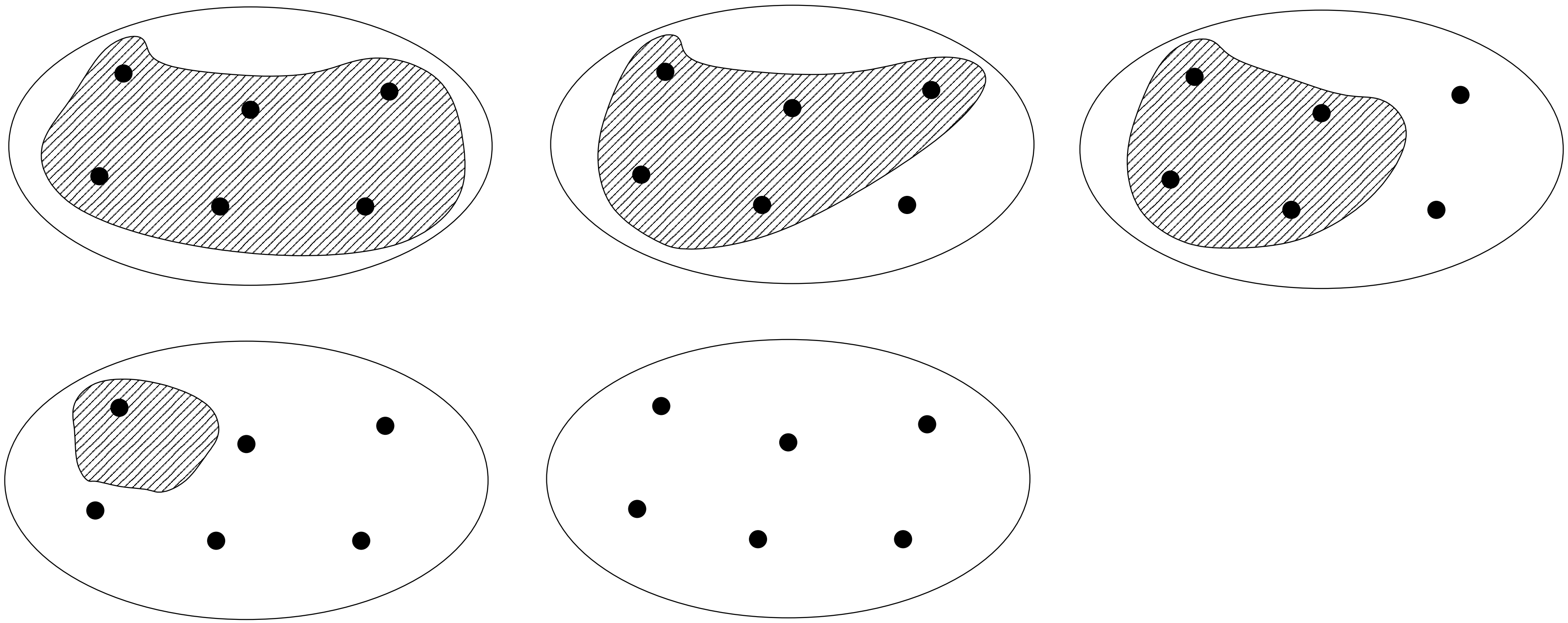
finite number $M_{\sigma,k}$ of k-equivalence classes of σ-structures

$$\Downarrow$$

$$\forall n \qquad N_{\sigma,k}(n) \leq M_{\sigma,k}$$

$\sigma = \{U\}$ $\quad k = 2$ $\quad n = 6$



$$N_{\{U\},2}(6) = 5$$

$$\forall n \geq 4 \qquad N_{\{U\},2}(n) = 5$$

Ash's Conjecture

$\forall\sigma\ \forall k \qquad n \mapsto N_{\sigma,k}(n)$ eventually **constant**

$\Downarrow \qquad$ (Ash 94)

Asser's Conjecture i.e. $NE = co - NE$

where $NE = \bigcup_{c\geq 1} NTIME(2^{c\cdot n})$

σ unary or $k=2$	$\sigma = \{E^{(2)}\}$?
$\Downarrow$	$k=3$?
$N_{\sigma,k}$ eventually constant	$\sigma = \{E^{(2)}\}$ and $k=3$?

Periodic Ash Conjecture	Ultra-weak Ash Conjecture
$\forall \sigma \ \forall k$	$\forall \sigma \ \forall k \ \forall i$
$N_{\sigma,k}$ eventually **periodic**	$N_{\sigma,k}^{-1}(i) \in NE$
$\Downarrow$ (Ash 94)	$\Updownarrow$ (CM06)
Asser's Conjecture	Asser's Conjecture

4.2 - Ash functions for classes of structures

$$\boxed{N_{\mathcal{C},k}}$$

σ relational signature

$\mathcal{C}$ class of σ-structures

k quantifier depth

n finite size

$N_{\mathcal{C},k}(n)$ = number of non k-equivalent structures in $\mathcal{C}$ of size n

$\mathcal{C}$ = Boolean algebras

$$N_{\mathcal{C},k}(n) = \begin{cases} 1 & \text{if } n = 2^m \\ 0 & \text{else} \end{cases}$$

$\mathcal{C}$ = equivalence relations?

$\mathcal{C}$ = bijections?

What makes these classes interesting?

Subcases of difficult cases with non-trivial expressive power :

$$\sigma = \{E_1^{(2)}, E_2^{(2)}\}$$

$\mathcal{C}_{2eqrel}$ = both equivalence relations $\quad$ $\mathcal{C}_{2funct}$ = both functions

(Any version of) Ash conjecture for

$$\forall k \quad N_{\mathcal{C}_{2funct},k} \qquad \text{or} \qquad \forall k \quad N_{\mathcal{C}_{2eqrel},k}$$

$$\Downarrow$$

Asser's Conjecture

4.3 - Application of characterization of k-equivalence for bijections

unbounded cycles w.r.t. k-equivalence

and *k-equivalence class* with or without unbounded cycles

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unbounded number of cycles w.r.t. k-equivalence

(with a given length $l \leq 2^{k-1}$)

and *k-equivalence class* with or without

unbounded number of cycles with a given length

Pumping lemma

k-equivalence class (containing arbitrarily large bijections)

has unbounded cycles

or

has unbounded number of cycles with some given lengths

---

for a given n every k-equivalence class

contributes for 0 or 1 to $N_{bij,k}(n)$

contribution to $N_{bij,k}$ of k-equivalence class

- *with* unbounded cycles is eventually constant 1
- *without* unbounded cycles is eventually periodic

$N_{bij,k}$ is eventually periodic

idem $N_{eq,k}$, $N_{funct,k}$, $N_{2eqsub,k}$, $N_{sg2,k}$, $N_{dg2,k}$, colored versions, ...

contribution to $N_{bij,k}$

of k-equivalence class without unbounded cycles

with unbounded number of cycles with a length $l_1, \ldots, l_r$

is 1 **exactly at** $\gcd(l_1, \ldots, l_r) \cdot \mathbb{N} + N_0$

Take **big** prime number $2^{k-2} + 1 \leq p \leq 2^{k-1}$

k-equivalence class $\mathcal{C}$ s.t. $\gcd(l_1, \ldots, l_r) = p$

$$\Downarrow$$

- $l_1 = \ldots = l_r = p$
- $\forall l \neq p$, number n_l fixed

All possible n_l for a k-equivalence class $\mathcal{C}$ s.t. $\gcd(l_1, \ldots, l_r) = p$

Length	1	2	3	4	$\ldots$	$2^{k-3}+1$	$\ldots$	2^{k-2}	$2^{k-2}+1$	$\ldots$	$p-1$	p	$p+1$	$\ldots$	2^{k-1}	$\geq 2^{k-1}+1$
Possible	0	0	0	0	$\ldots$	0	$\ldots$	0	0	$\ldots$	0	∞	0	$\ldots$	0	0
values	$\vdots$	$\vdots$	$\vdots$	$\vdots$		1	$\ldots$	1								
for	$\vdots$	$\vdots$	$k-3$	$k-3$												
n_l	$\vdots$	$k-2$														
	$k-1$															

total number of k-equivalence classes s.t. $\gcd(l_1, \ldots, l_r) = p$

$$k \cdot \prod_{i=1}^{k-2} (k-i)^{2^{i-1}} \ \textbf{not} \text{ divisible by } p$$

non constant overall contribution to $N_{bij,k}$

of k-equivalence classes s.t. $\gcd(l_1, \dots, l_r) = p$

---

not offset by contributions of other k-equivalence classes

(periodic with a period $d \perp p$)

Theorem (CM)

For all $k \geq 3$ the Ash functions $N_{bij,k}$ are **not** eventually constant