A Rank Technique for Formula Size Lower Bounds

Troy Lee

LRI, Université Paris-Sud

Formula size

- Weakening of the circuit model—a formula is a binary tree with internal nodes labeled by AND, OR and leaves labeled by literals. The size of a formula is its number of leaves.

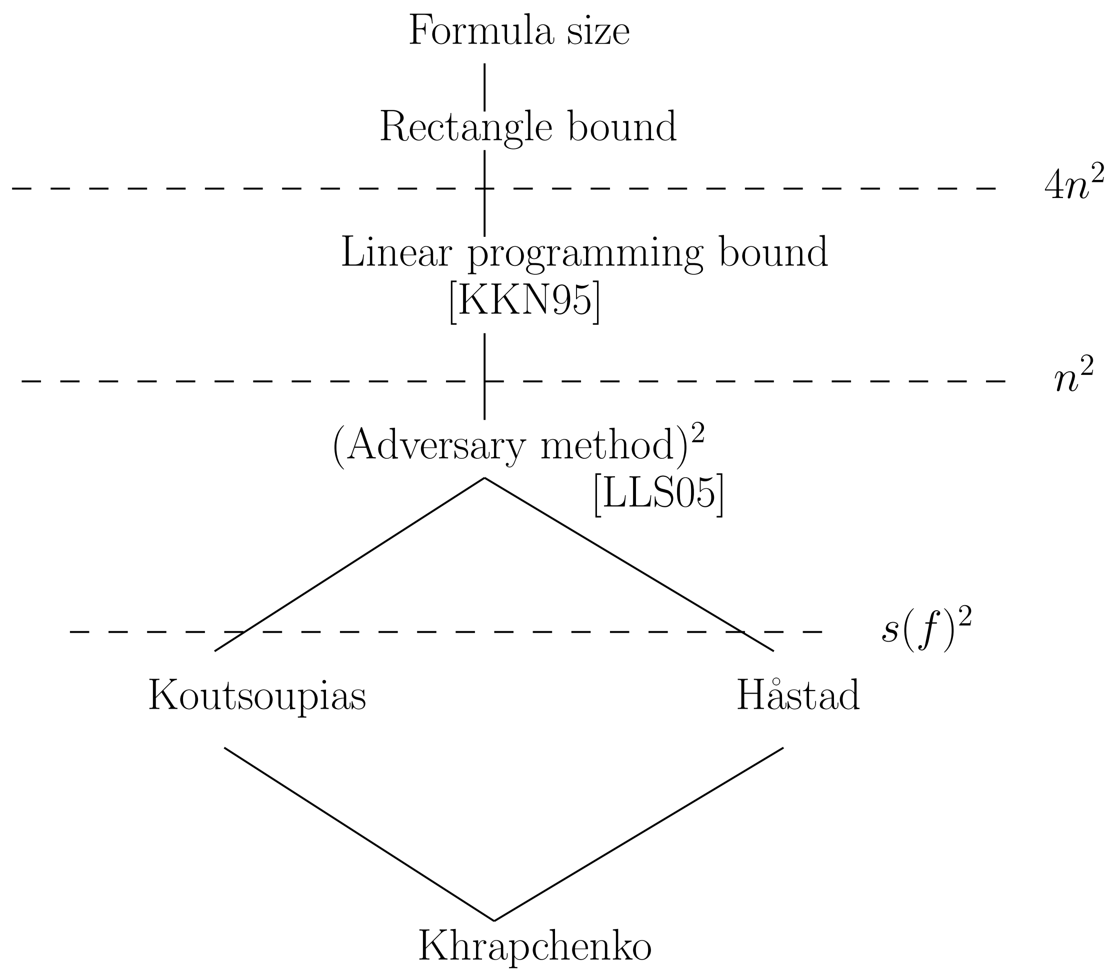
- PARITY has formula size $\theta(n^2)$ [Khr71].

- The best lower bound for a function in NP is $n^{3-o(1)}$ [Hås98].

- Many open questions:

$$\frac{n^2}{4} \leq L(\text{MAJORITY}) \leq n^{4.57}$$

The n^2 barrier. . .



A new rank technique

- We devise a new lower bound technique based on matrix rank.

- We exactly determine the formula size of PARITY: if $n = 2^\ell + k$ then

$$L(\text{PARITY}) = 2^\ell(2^\ell + 3k) = n^2 + k2^\ell - k^2.$$

- Bound can be viewed as a simultaneous rigidity problem

Karchmer–Wigderson game [KW88]

- Elegant characterization of formula size in terms of a communication game.

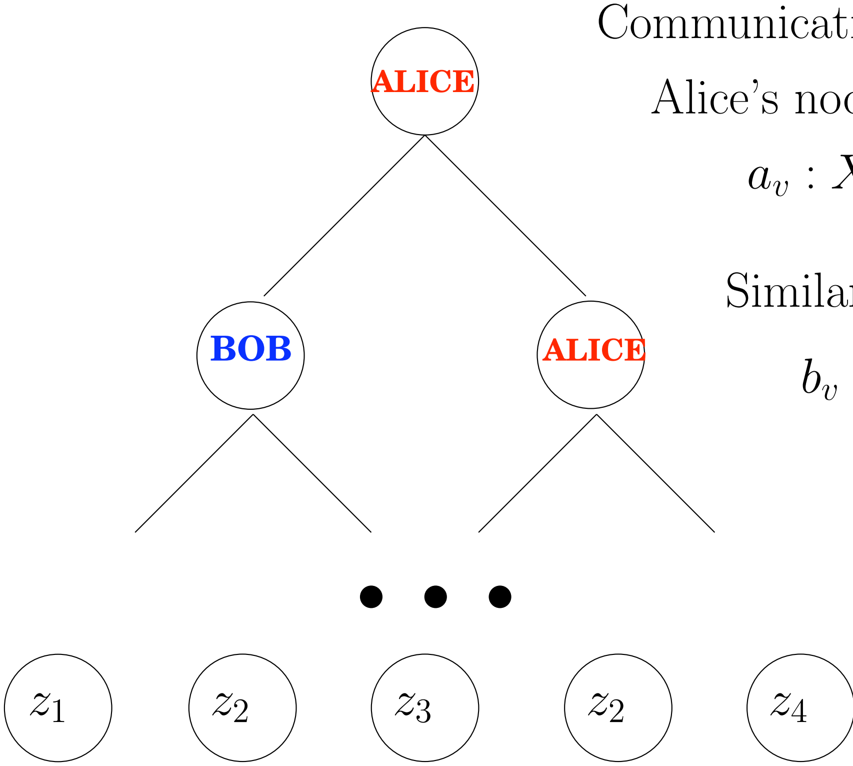
- For a Boolean function f, let $X = f^{-1}(0), Y = f^{-1}(1)$ and

$$R_f = \{(x, y, i) : x \in X,\ y \in Y,\ x_i \neq y_i\}$$

- The game is then the following: Alice is given $x \in X$, Bob is given $y \in Y$ and they wish to find i such that $(x, y, i) \in R_f$.

- Karchmer–Wigderson Thm: The number of leaves in a best communication protocol for R_f equals the formula size of f.

Communication complexity of relations
$R \subseteq X \times Y \times Z$



Communication protocol is a binary tree:

Alice's nodes labelled by a function:

$a_v : X \longrightarrow \{0, 1\}$

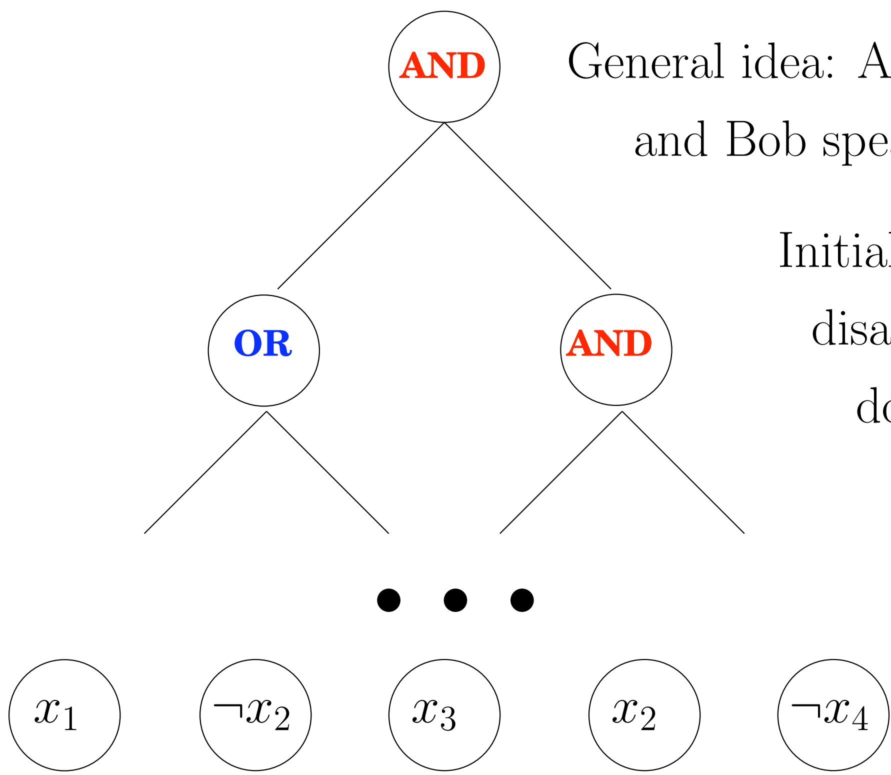
Similarly, Bob's nodes labelled

$b_v : Y \longrightarrow \{0, 1\}$

Leaves labelled by elements $z \in Z$.

Denote by $C^P(R)$ the number of leaves in a best protocol for R.

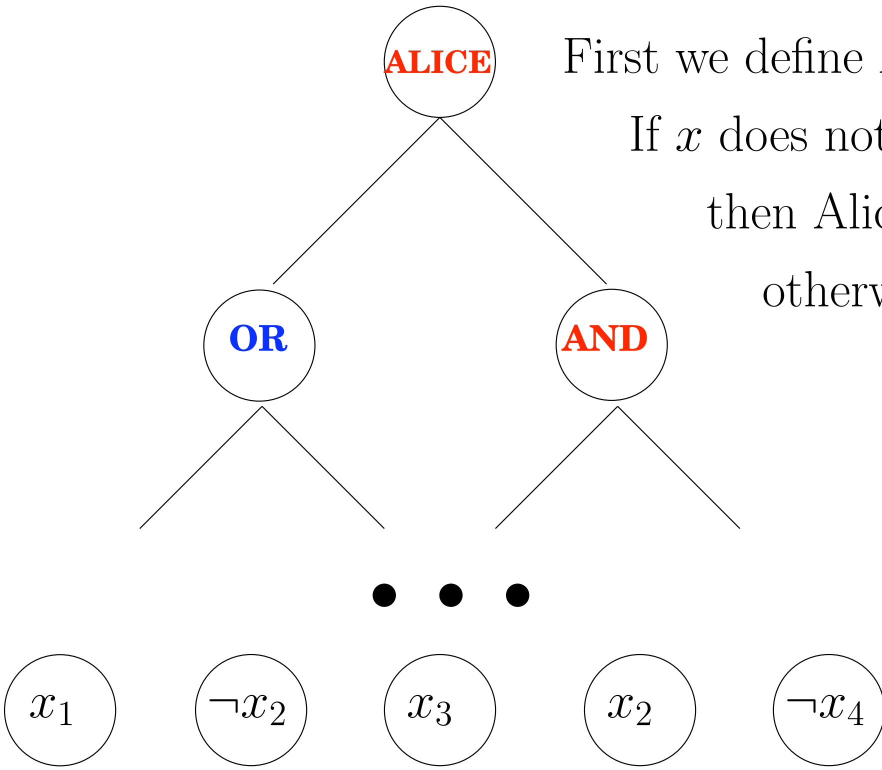
Proof by picture: $C^P(R_f) \leq \mathsf{L}(f)$.



General idea: Alice speaks at AND nodes and Bob speaks at OR nodes.

Initially, $f(x) \neq f(y)$ and we maintain this disagreement on subformulas as we move down the tree.

Proof by picture: $C^P(R_f) \leq \mathsf{L}(f)$.



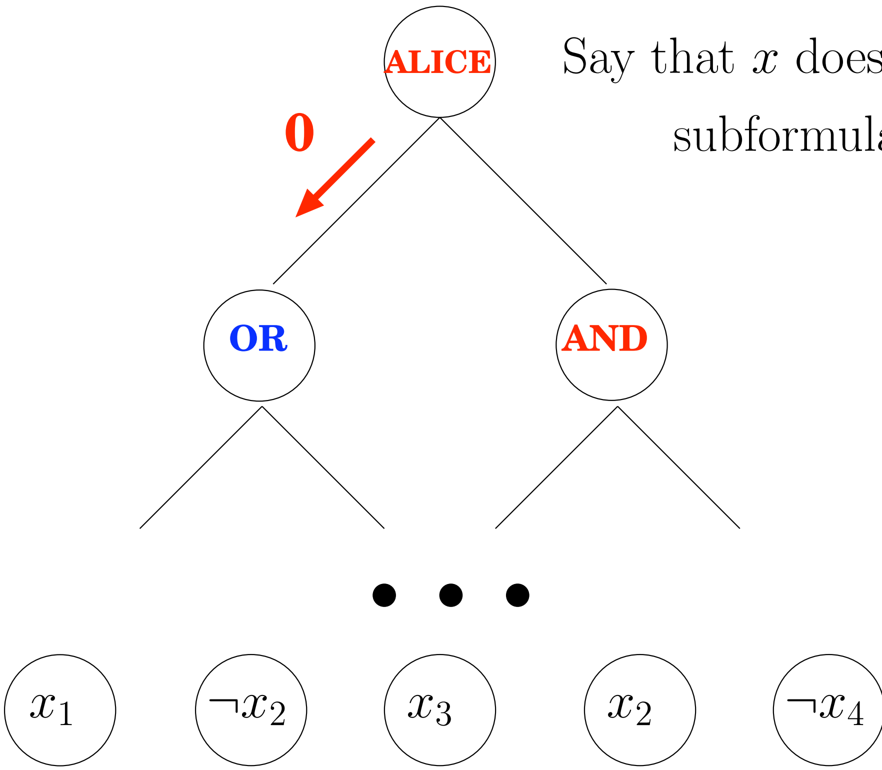
First we define Alice's action at the top node:

If x does not satisfy the left subformula,

then Alice sends the bit 0;

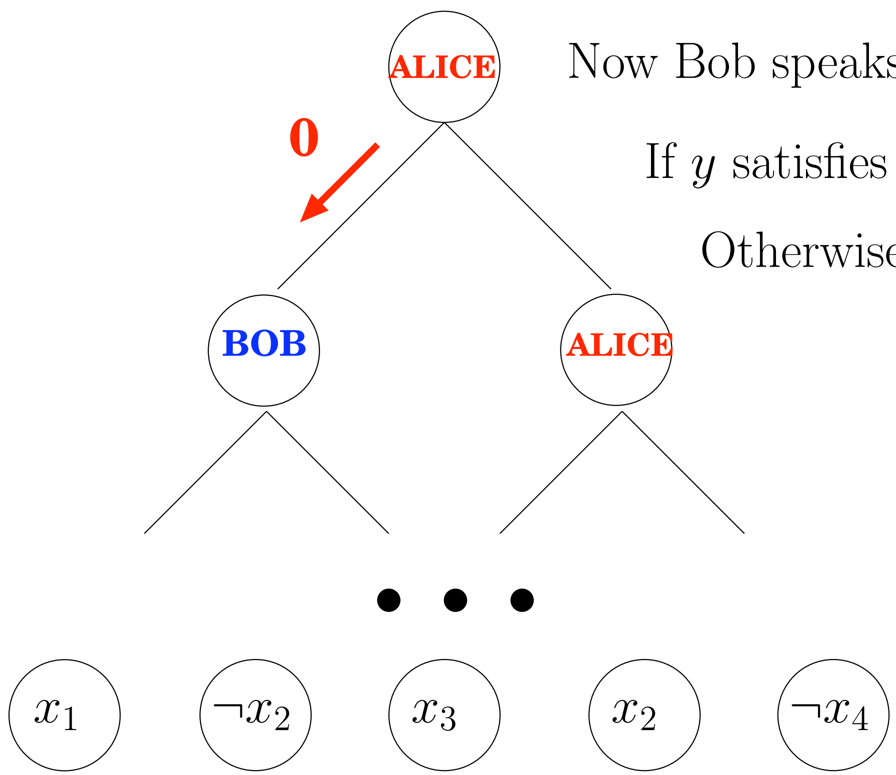
otherwise she sends the bit 1.

Proof by picture: $C^P(R_f) \leq \mathsf{L}(f)$.



Say that x does not satisfy the left subformula.

Proof by picture: $C^P(R_f) \le \mathsf{L}(f)$.

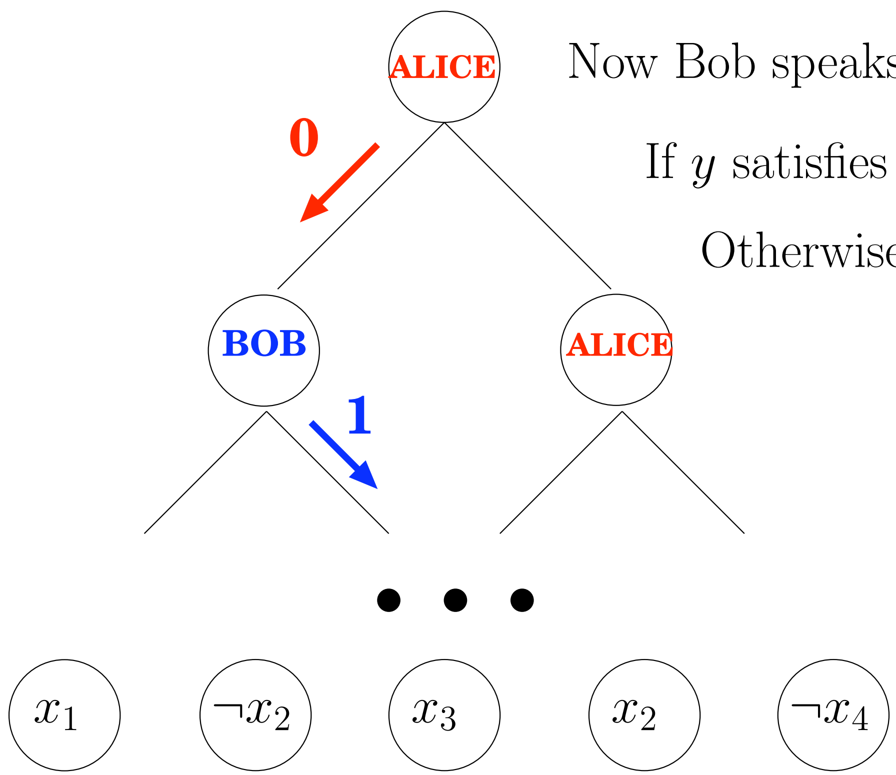


Now Bob speaks at the OR gate:

If y satisfies the left subformula, Bob says 0.

Otherwise, he says 1.

Proof by picture: $C^P(R_f) \leq \mathsf{L}(f)$.

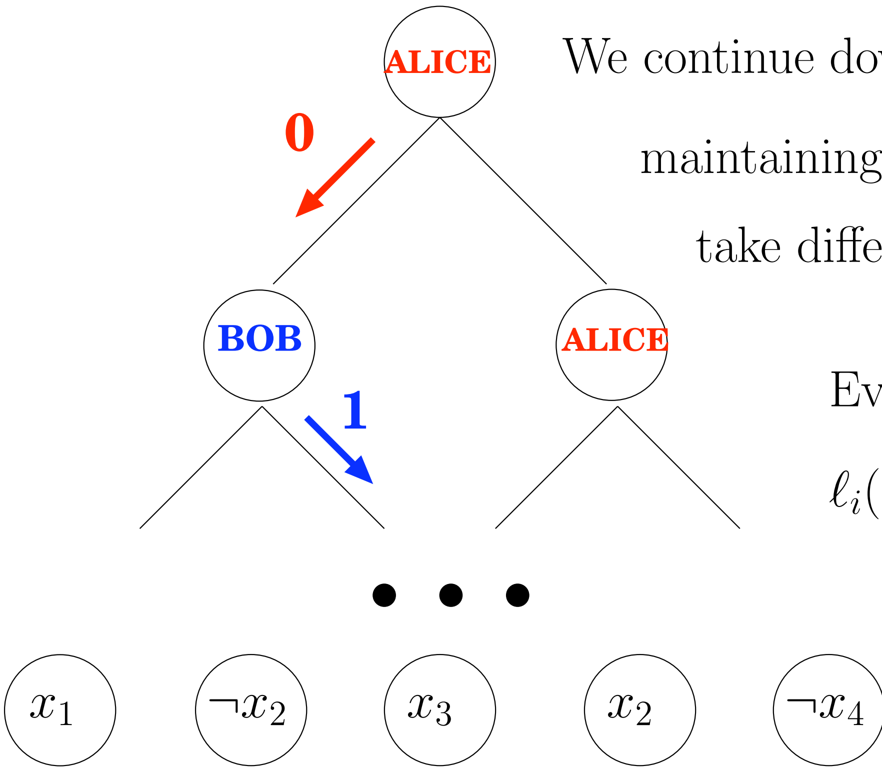


Now Bob speaks at the OR gate:

If y satisfies the left subformula, Bob says 0.

Otherwise, he says 1.

Proof by picture: $C^P(R_f) \leq \mathsf{L}(f)$.



We continue down the tree in a similar fashion,

maintaining the property that x and y

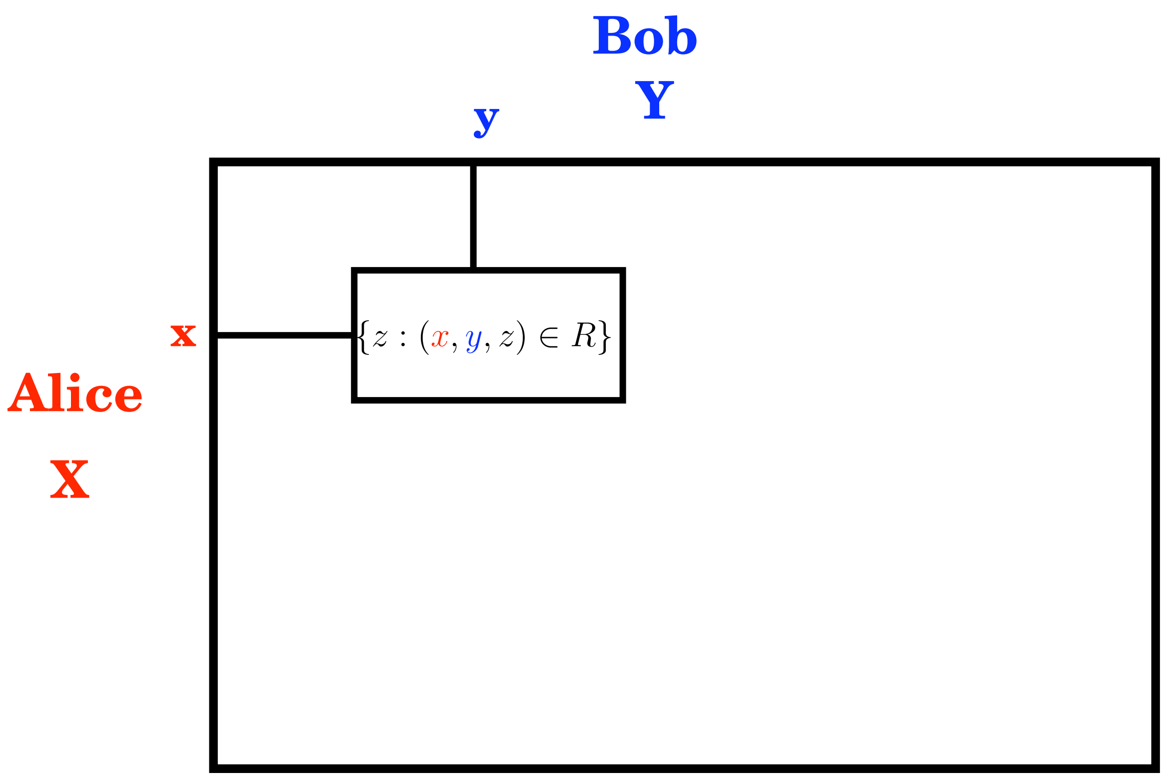
take different values on subformulas.

Eventually, we reach a literal ℓ_i such that

$\ell_i(x) \neq \ell_i(y)$ and so x and y differ on bit i.

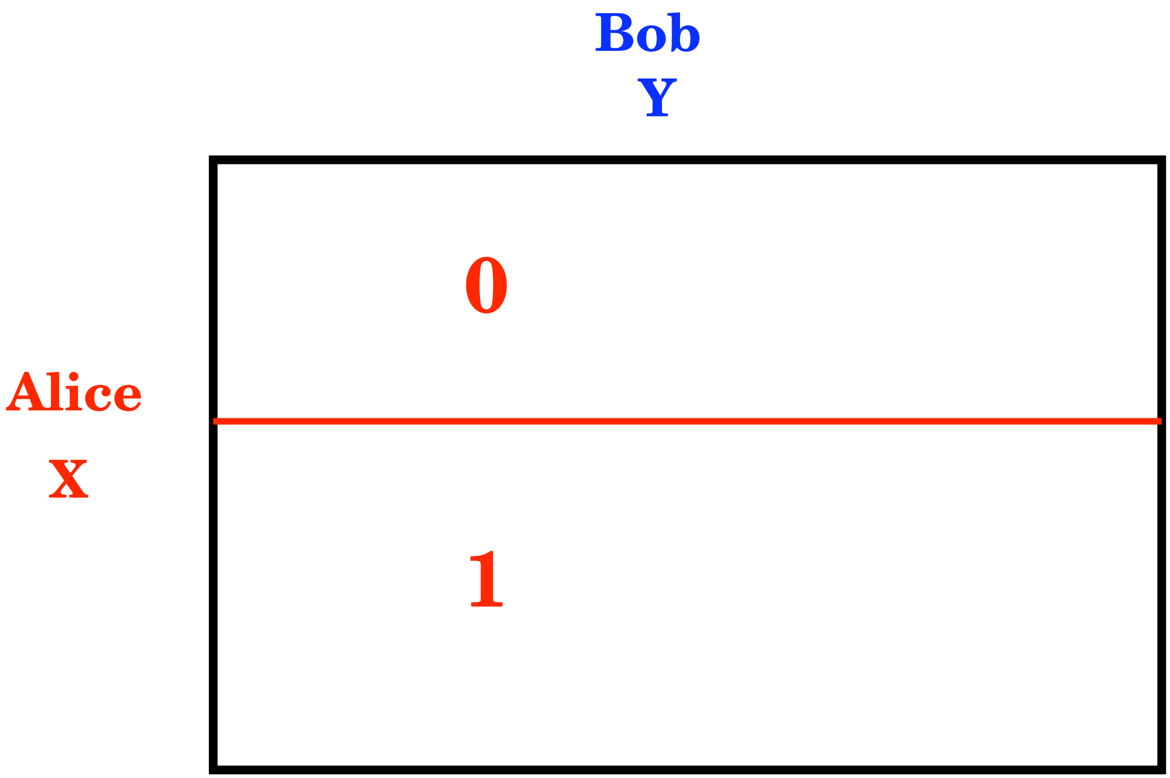
Communication Complexity and the Rectangle Bound

$R \subseteq X \times Y \times Z$



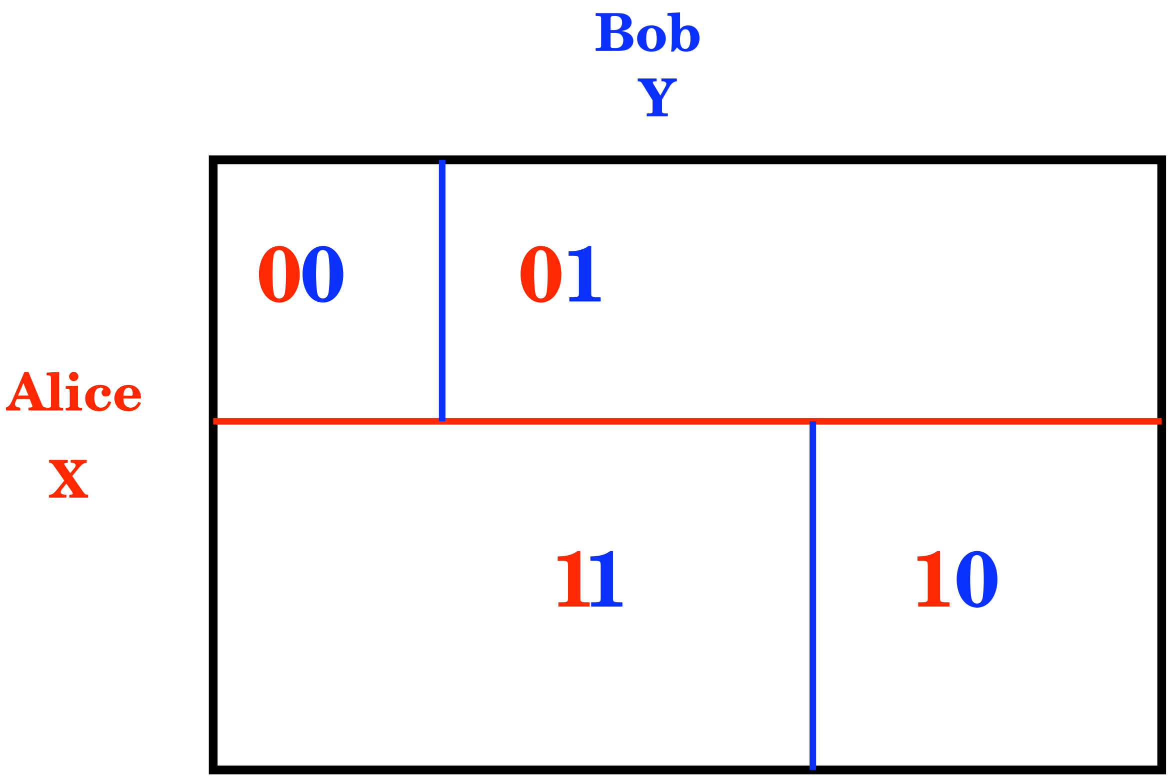
Communication Complexity and the Rectangle Bound

$$R \subseteq X \times Y \times Z$$



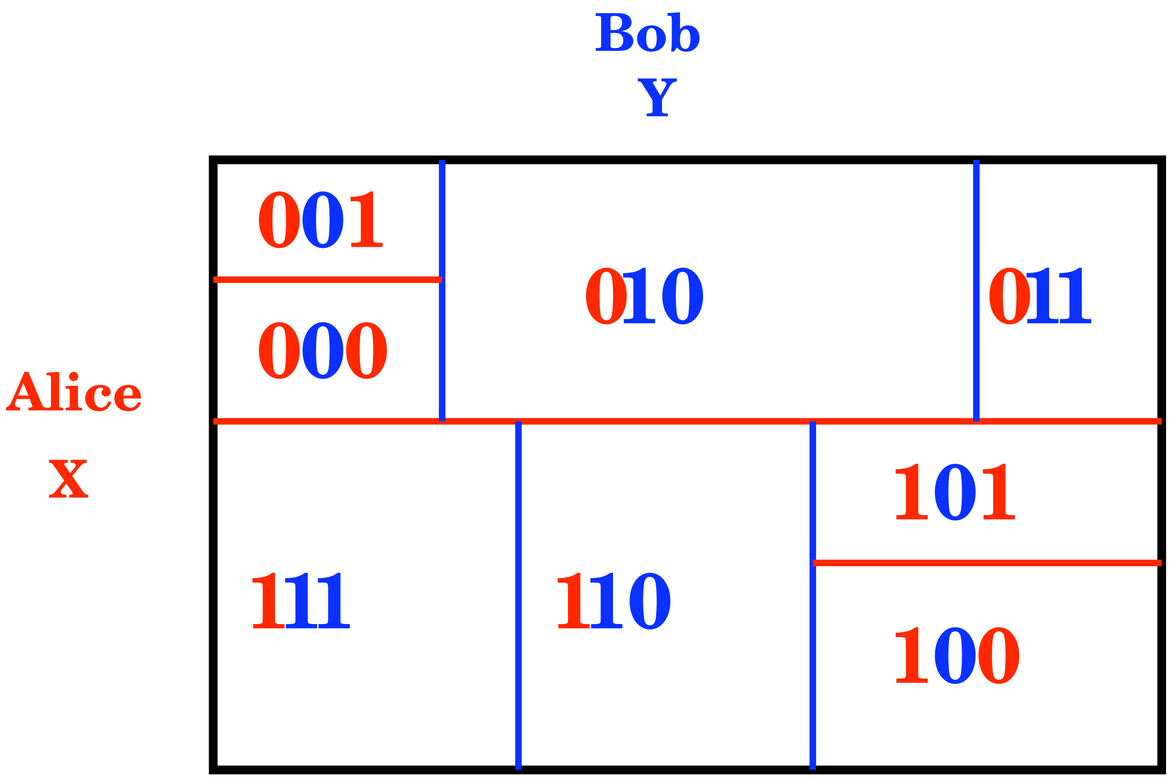
Communication Complexity and the Rectangle Bound

$R \subseteq X \times Y \times Z$



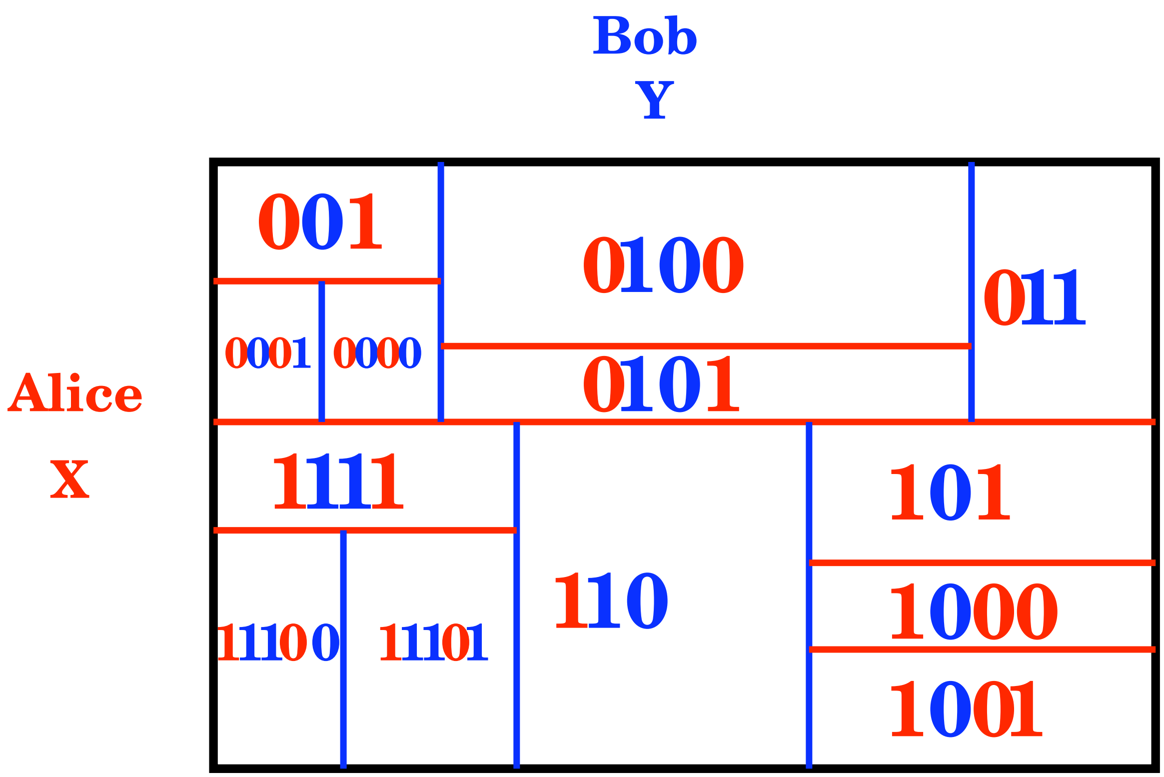
Communication Complexity and the Rectangle Bound

$R \subseteq X \times Y \times Z$



Communication Complexity and the Rectangle Bound

$R \subseteq X \times Y \times Z$



A rectangle S is monochromatic if there exists z such that $(x, y, z) \in S$ for all $(x, y) \in S$.

A successful protocol partitions $X \times Y$ into monochromatic rectangles.

Communication Complexity and the Rectangle Bound

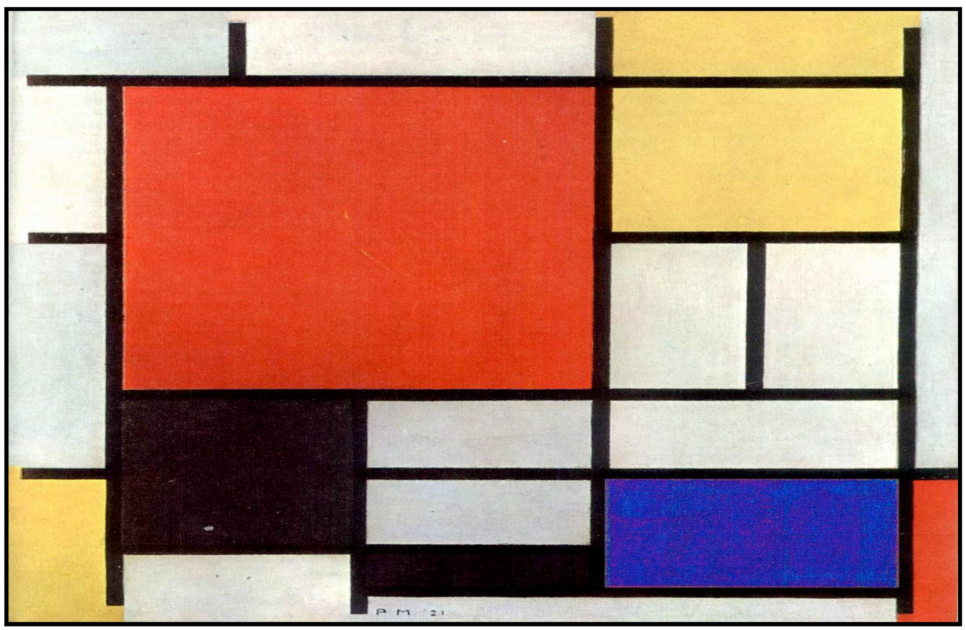
$R \subseteq X \times Y \times Z$

Bob

Y

Alice

X



Rectangles and rank

- Rank is one of the most successful ways to prove lower bounds on communication complexity of functions.

- Let $M_f[x, y] = f(x, y)$. A monochromatic 1-rectangle has rank one, $\text{rk}(M_f) \leq C^P(f)$.

- Rank conjectured to be nearly tight: $\log C^P(f) \leq (\log \text{rk}(M_f))^{O(1)}$.

- It has been difficult to adapt the rank technique to communication complexity of relations.

Rank for relations

- The key idea is a selection function $S : X \times Y \rightarrow Z$.

- A selection function turns a relation into a function, by selecting one valid output.

- Let $R|_S = \{(x, y, z) : S(x, y) = z\}$. Then
$$C^P(R) = \min_S C^P(R|_S).$$

Rank for relations

- With the help of selection functions, we can now apply the rank method as before.

- Let S_z be a matrix where $S_z[x, y] = 1$ if $S(x, y) = z$ and 0 otherwise.

$$\min_S \sum_{z \in Z} \mathrm{rk}(S_z) \leq \min_S C^P(R|_S) = C^P(R)$$

A rigidity type problem

Given an identity matrix, what is fastest way to reduce rank by flipping zeros to ones?

$$
\begin{matrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{matrix}
$$

A rigidity type problem

Given an identity matrix, what is fastest way to reduce rank by flipping zeros to ones?

$$\begin{matrix} 1 & \color{red}{1} & 0 & 0 & 0 & 0 \\ \color{red}{1} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{matrix}$$

A rigidity type problem

Given an identity matrix, what is fastest way to reduce rank by flipping zeros to ones?

$$\begin{matrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{matrix}$$

A rigidity type problem

Given an identity matrix, what is fastest way to reduce rank by flipping zeros to ones?

$$
\begin{matrix}
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1
\end{matrix}
$$

This construction is optimal

- To reduce rank to n/k need to add $n(k-1)$ many ones.

This can be seen as, by Cauchy-Schwarz,

$$\left\lceil \frac{\|M\|_{\mathrm{tr}}^2}{\|M\|_F^2} \right\rceil \leq \mathrm{rk}(M)$$

where

- $\|M\|_{\mathrm{tr}} = \sum_i \lambda_i(M)$
- $\|M\|_F^2 = \sum_{x,y} M[x,y]^2 = \sum_i \lambda_i^2(M)$

Simple form of rank bound

- Let $S : X \times Y \to [n]$ be a selection function, and let s_i be number of entries (x, y) where S chooses i.

- Applying lower bound on rank:

$$\min_{s_i} \sum_i \left\lceil \frac{\|S_i\|_{\mathsf{tr}}^2}{s_i} \right\rceil \leq \min_S \sum_i \mathsf{rk}(S_i) \leq C^P(R).$$

- Simple form depends not on *which* pairs (x, y) are are selected to output i, but only on number.

Application to Parity

- Selection function: $S : 2^{n-1} \times 2^{n-1} \to [n]$.

- For every $i \in [n]$, there are 2^{n-1} pairs where behavior of selection function is determined—the sensitive pairs.

- If selection function S only output i where forced to, then $\mathsf{rk}(S_i) = 2^{n-1}$. Thus S must output i in more places to bring down rank.

Application to Parity

- Because of sensitive pairs $\|S_i\|_{\text{tr}} \geq 2^{n-1}$ for every i.

- We have the bound
$$\min_{s_i} \sum_i \left\lceil \frac{(2^{n-1})^2}{s_i} \right\rceil \leq L(\text{PARITY})$$
where $\sum_i s_i = (2^{n-1})^2$.

- Ignoring the ceilings, Jensen's inequality says minimum attained when all s_i equal, $s_i = (2^{n-1})^2/n$, which gives bound n^2.

Application to Parity

Recall our bound:

$$\min_{s_i} \sum_i \left\lceil \frac{(2^{n-1})^2}{s_i} \right\rceil \leq L(\text{PARITY})$$

where $\sum_i s_i = (2^{n-1})^2$.

If n is not a power of two, cannot take all s_i equal. If $n = 2^\ell + k$, best thing to do: take each s_i a power of two, as evenly as possible:

$$L(\text{PARITY}) = 2^\ell(2^\ell + 3k) = n^2 + k2^\ell - k^2$$

Upper bound

- In upper bound, we have exactly the same optimization

- Basic idea: binary search. Alice says parity of left half of x, Bob says parity of left half of y. Continue on half which disagree.

- When n is not power of 2, balance search tree as evenly as possible.

Open problems

- Application to threshold functions?

$$\frac{n^2}{4} \leq L(\text{MAJORITY}) \leq n^{4.57}$$

- More subtle lower bound on rank? Use not just number of ones in each S_i but also their placement...