Model Theory on Well-Behaved Classes of Finite Structures

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Finite Model Theory – Early Days

In the 1980s, the term *finite model theory* came to be used to describe the study of the expressive power of logics (from first-order to second-order logic and in between), on the class of all finite structures.

The motivation for the study is that problems in computer science (especially in *complexity theory* and *database theory*) are naturally expressed as questions about the expressive power of logics.

And, the structures involved in computation are finite.

Finite Model Theory – Early Trends

Kolaitis (LICS 93) identified trends in the results in finite model theory.

- *Negative*: showing the failure of classical model-theoretic results on finite structures.

  Compactness. Completeness. Interpolation and preservation theorems.

- *Conservative*: showing that certain classical model-theoretic results continue to hold on finite structures.

  Some consequences of compactness. Monotone vs. positive inductions. Locality.

- *Positive*: exploring concepts and results which are specific to finite structures.

  Descriptive complexity. 0–1 laws.

Descriptive Complexity

A class of finite structures is definable in existential second-order logic if, and only if, it is decidable in NP.

(Fagin)

A class of *ordered* finite structures is definable in least fixed-point logic if, and only if, it is decidable in P.

(Immerman; Vardi)

Open Question: Is there a logic that captures P without order?

LFP+counting does not suffice.

(Cai, Fürer and Immerman)

Is Finiteness Natural?

The finite structures are not a natural class.

Jon Barwise

While *compactness* cannot be recovered without allowing some infinite structures, some of its consequences can be, on restricted classes of finite structures.

Often, the structures of interest in computer science do not form the class of *all* finite structures, but some subclass.

How does such a restriction affect the model-theoretic results?

Existential Preservation

A sentence φ is equivalent to an existential sentence if, and only if, the models of φ are closed under extensions.

(Łoś-Tarski)

There is a sentence φ that is preserved under extensions on the class of all finite structures, but is not equivalent *on finite structures* to an existential sentence.

(Tait)

Restrict to a class of finite structures $\mathcal{C}$.

If φ is preserved under extensions on $\mathcal{C}$, is it equivalent on $\mathcal{C}$ to an existential sentence?

Atserias, D., Grohe, 2005

Homomorphism Preservation

A sentence φ is equivalent to an *existential positive* sentence if, and only if, the models of φ are closed under homomorphisms.

A sentence φ is equivalent to an existential positive sentence on finite structures if, and only if, the finite models of φ are closed under homomorphisms.

(Rossman 05)

Restrict to a class of finite structures $\mathcal{C}$.

If φ is preserved under homomorphisms on $\mathcal{C}$, is it equivalent on $\mathcal{C}$ to an existential positive sentence?

Atserias, D., Kolaitis, 2004

Restricted Classes

In this talk, we look at classes of finite structures restricted by the form of their *adjacency* (or *Gaifman*) graph.

> This is the graph on the universe of the structure where two nodes are adjacent if they appear together in some relation.

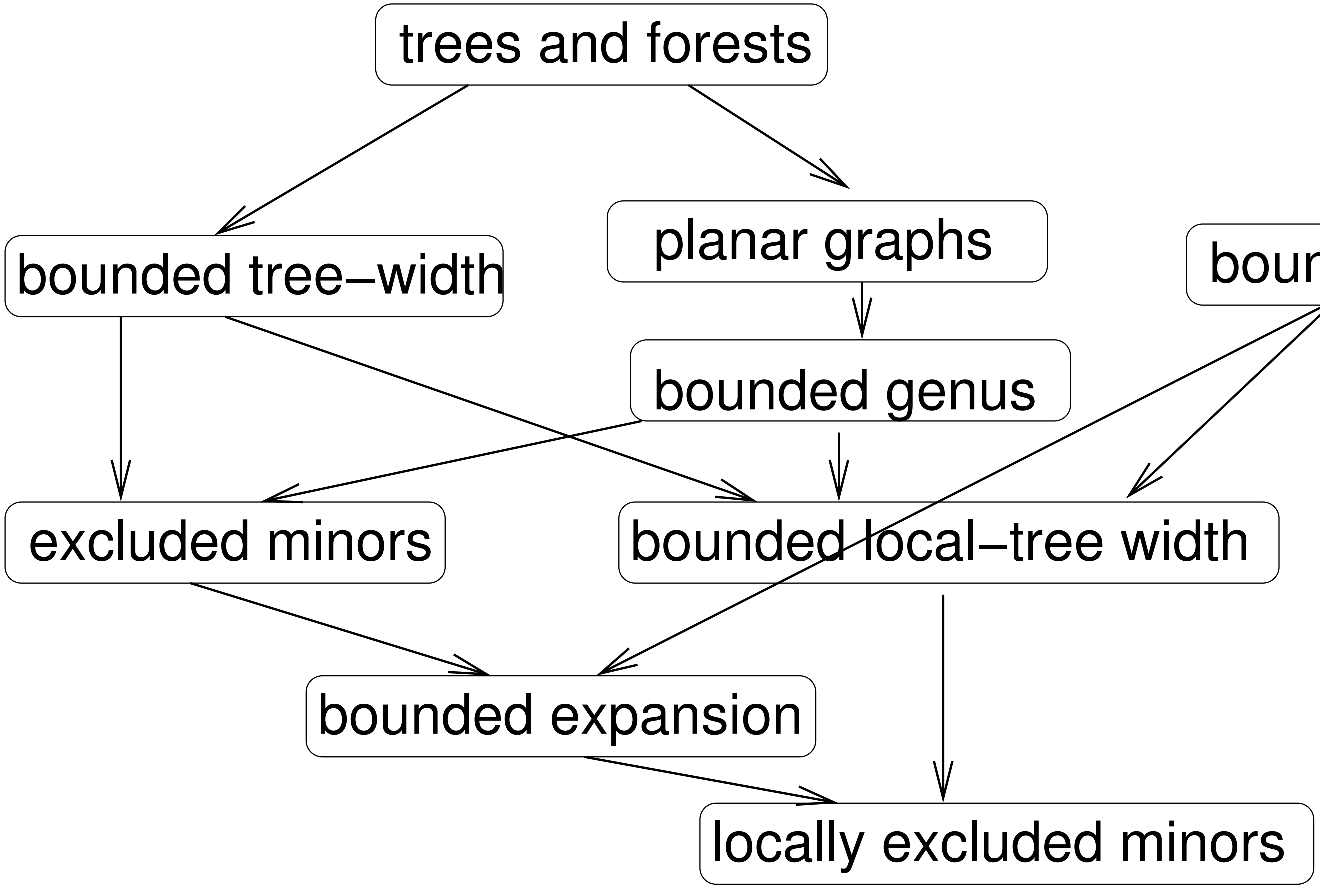
The classes are based on classes of graphs with good *algorithmic* properties.

Do they also have good *model-theoretic* properties?

While this talk focuses on preservation properties, many other natural model-theoretic questions arise.

Well-Behaved Classes of Finite Structures

Certain classes of finite structures have been recognized as *well-behaved* (to different degrees):



Well-Behavedness

Often, the good algorithmic properties of a class are explained by, or related to, a logical result.

On any class of structures of bounded tree-width, $\mathbb{A} \models \varphi$ for an MSO formula φ is decidable in time $f(\varphi)O(|\mathbb{A}|)$. (MSO is fixed-parameter tractable on classes of bounded tree-width.)

(Courcelle 1990).

Any first-order definable set-optimisation problem has a *polynomial approximation scheme* on any class of graphs that excludes a minor.

(D., Grohe, Kreutzer, Schweikardt 2006).

LFP+counting captures P on the class of planar graphs.

(Grohe 2000).

Trees and Forests

- Every order-invariant first-order formula is equivalent to one without order on trees. (Benedikt-Segoufin 06).
- In general, interpolation fails.
- MSO validity is decidable and model-checking is fixed-parameter tractable.
- The extension-preservation property holds. (Atserias, D., Grohe 05)
- The homomorphism-preservation property holds. (Atserias, D., Kolaitis 04)
- LFP+counting captures P. (Immerman&Lander; Lindell 92).

Some of these properties, but not all, are inherited by sub-classes.

Minimal Models

If φ is a first-order sentence whose models are closed under extensions, then we say that $\mathbb{A}$ is a *minimal* model of φ if,

$\mathbb{A} \models \varphi$ and no proper *induced* substructure $\mathbb{B}$ of $\mathbb{A}$ is a model of φ

A sentence φ that is invariant under extensions is equivalent to an existential sentence if, and only if, it has finitely many minimal models.

To prove the extension preservation theorem on a class of finite structures $\mathcal{C}$, we aim to show that for any sentence φ, there is an N such that all minimal models of φ in $\mathcal{C}$ have at most N elements.

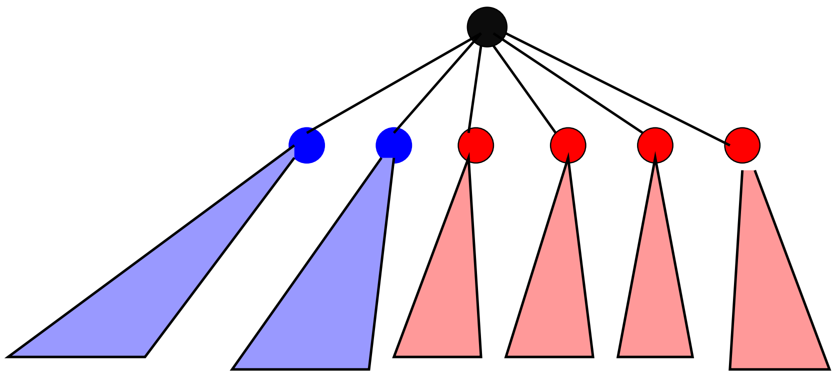
Extension Preservation on Forests

Theorem

The extension preservation property holds in the class of finite forests.

Let φ with quantifier rank m be closed under extensions in this class.

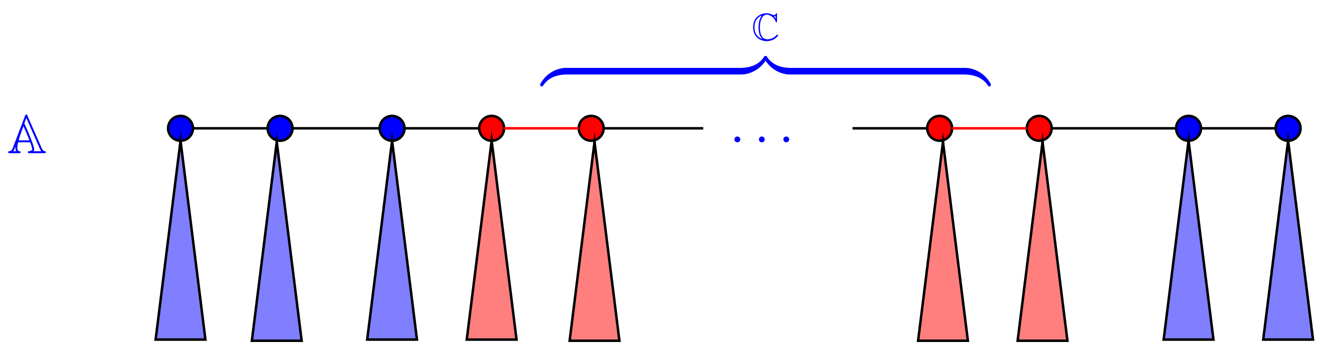
As $\equiv_m$ has finite index, a minimal model of φ cannot have arbitrarily large degree.



If there are m copies of $\equiv_m$-equivalent subtrees, delete one to get a proper submodel.

Extension Preservation on Forests

Also, a minimal model of φ cannot have an arbitrarily long path.



$\mathbb{C}$ is a segment of frequently occurring types of subtree.

$$\mathbb{A} - \mathbb{C} \equiv_m \mathbb{A} + \mathbb{C}$$

The extension preservation property holds in any class of acyclic structures closed under substructures and disjoint unions.

Note: quadruple-exponential blow-up in formula size.

Tree-Width

Tree-width is a measure of how *tree-like* a structure is.

For a graph $G = (V, E)$, a *tree decomposition* of G is a relation $D \subset V \times T$ with a tree T such that:

- for each $v \in V$, the set $\{t \mid (v, t) \in D\}$ forms a connected subtree of T; and
- for each edge $(u, v) \in E$, there is a $t \in T$ such that $(u, t), (v, t) \in D$.

The *tree-width* of G is the least k such that there is a tree T and a tree-decomposition $D \subset V \times T$ such that for each $t \in T$,

$$|\{v \in V \mid (v, t) \in D\}| \leq k + 1.$$

Examples

- Trees have tree-width 1.
- Cycles have tree-width 2.
- The clique K_k has tree-width $k - 1$.
- The $m \times n$ grid has tree-width $\min(m, n)$.

Bounded Tree-Width

Let $\mathcal{T}_k$ be the class of structures of tree-width at most k.

- Order-invariance?
- MSO validity is decidable and satisfaction is fixed-parameter tractable. (Seese; Courcelle).
- The extension-preservation property holds. (Atserias, D., Grohe)
- The homomorphism-preservation property holds. (Atserias, D., Kolaitis)
- LFP$+$counting captures P. (Grohe, Mariño).

There is a (sort of) converse to the last statement.

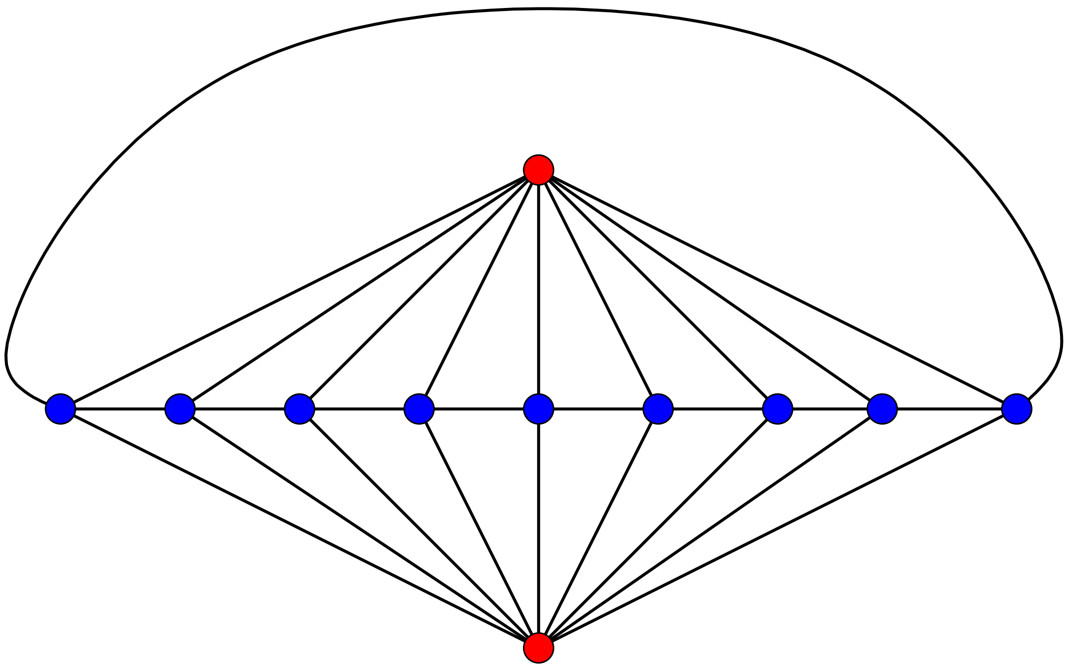
LFP$+$counting captures P on $\mathcal{T}_{f(n)}$ only if f is constant. (D., Richerby)

Bounded Tree-width

Let $\mathcal{C}$ be a subclass of $\mathcal{T}_k$.

- The extension-preservation property fails, in general. In particular, it fails on the class of planar graphs of treewidth at most 4. (Atserias, D., Grohe)
- The homomorphism-preservation property holds. (Atserias, D., Kolaitis)

Extension Preservation Fails on Planar Graphs



“There are two red vertices such that *if* every other vertex is a neighbour of both *then* every vertex has at least two blue neighbours”.

Bounded-Degree Structures

- MSO validity is undecidable and satisfaction is intractable.
- FO satisfaction is fixed-parameter tractable. (Seese 96).
- The extension-preservation property holds. (Atserias, D., Grohe 05)
- The homomorphism-preservation property holds. (Atserias, D., Kolaitis 04)
- A logic capturing P?

Wide Classes

Definition

A class of structures $\mathcal{C}$ is said to be *wide* if for every d and m there is an N such that any structure in $\mathcal{C}$ with more than N elements contains a d-scattered set of size m.

Example: Classes of structures of bounded degree.

Definition

A class of structures $\mathcal{C}$ is *almost wide* if there is an s such that for every d and m there is an N such that any structure in $\mathcal{C}$ with more than N elements contains s elements whose removal leaves a d-scattered set of size m.

Example: Trees.

Preservation on Wide Classes

The extension preservation theorem holds in any class $\mathcal{C}$ that is

- *wide*
- closed under taking substructures
- closed under disjoint unions

(Atserias, D., Grohe 2005).

The homomorphism preservation theorem holds in any class $\mathcal{C}$ that is

- *almost wide*
- closed under taking substructures
- closed under disjoint unions

(Atserias, D., Kolaitis 2004) essentially based on (Ajtai, Gurevich 1994).

Gaifman Locality

Gaifman locality is a key ingredient of the proofs.

A basic local formula $\psi^r(x)$ is a formula in which all quantifiers are relativised to $\mathrm{Nbd}^r(x)$.

A basic local sentence is one of the form

$$\exists x_1 \ldots \exists x_n \bigwedge_{i \neq j} \mathrm{Nbd}^r(x_i) \cap \mathrm{Nbd}^r(x_j) = \emptyset \wedge \bigwedge_i \psi^r(x_i)$$

Theorem (Gaifman)

Every first-order sentence is equivalent to the Boolean combination of basic local sentences.

Homomorphism Preservation on Wide Classes

Show that if φ is preserved homomorphisms, then a minimal model cannot have a large scattered set.

If $\mathbb{A}$ contains a large enough scattered set, it contains two elements a and a' such that the basic local formulas (up to some suitable quantifier rank) satisfied in $\mathrm{Nbd}^r(a)$ and $\mathrm{Nbd}^r(a')$ are the same.

Let $\mathbb{A}'$ be $\mathbb{A}$ with a removed.

$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{\text{hom}} & \mathbb{A} + n \cdot \mathbb{A}' \\ & & \equiv_q \\ \mathbb{A}' & \xleftarrow{\text{hom}} & n \cdot \mathbb{A}' \end{array}$$

Extension Preservation on Wide Classes

The proof of *extension preservation* is more involved, but again relies on Gaifman locality.

In any model $\mathbb{A}$ with large enough scattered sets, we find a substructure $\mathbb{A}'$ and an extension $\mathbb{B}$ of $\mathbb{A}$ such that

$$\mathbb{A}' \equiv_m \mathbb{B}.$$

We construct $\mathbb{A}'$ in a series of stages by including all *neighbourhoods* of *rare type* and certain neighbourhoods of frequent type.

$\mathbb{B}$ is then obtained as the disjoint union of $\mathbb{A}$ with the neighbourhoods in $\mathbb{A}'$ of frequent type.

The *types* used are MSO types of neighbourhoods.

Excluded Minor Classes

- FO is fixed-parameter tractable. (Flum, Grohe).
- The extension-preservation property fails, even on planar graphs. (Atserias, D., Grohe)
- The homomorphism-preservation property holds. (Atserias, D., Kolaitis)
- LFP+counting captures P on graphs of bounded genus. (Grohe).
- A logic capturing P on excluded minor classes?

Graph Minors

We say that a graph $G = (V, E)$ is a minor of graph $H = (U, F)$, (written $G \prec H$) if there is a graph $H' = (U', F')$ with $U' \subseteq U$ and $F' \subseteq F$ and a surjective map

$$M : U' \to V$$

such that

- for each $v \in V$, $M^{-1}(v)$ is a connected subgraph of H'; and
- for each edge $(u, v) \in E$, there is an edge in F' between some $x \in M^{-1}(u)$ and some $y \in M^{-1}(v)$.

Facts about Graph Minors

G is planar if, and only if, $K_5 \not\prec G$ and $K_{3,3} \not\prec G$.

If $G \prec H$, then $\text{tree-width}(G) \leq \text{tree-width}(H)$.

The relation $\prec$ is transitive.

If $\text{tree-width}(G) < k - 1$, then $K_k \not\prec G$.

$K_k \prec K_{k-1,k-1}$.

A class of graphs $\mathcal{C}$ has bounded treewidth if, and only if, there is some grid G such that $G \not\prec H$ for any $H \in \mathcal{C}$.

Theorem (Robertson-Seymour)

In any infinite collection $\{G_i \mid i \in \omega\}$ of graphs, there are i, j with $G_i \prec G_j$.

Excluded Minor Classes are Almost Wide

The main combinatorial construction in (Atserias, D., Kolaitis) (based on (Kreidler and Seese 1999)) shows that if $\mathcal{C}$ *excludes a graph minor* then $\mathcal{C}$ is almost wide.

I.e.:

> For each k, d and m there is an N such that if G is a graph with $K_k \not\preceq G$, and $|G| > N$, then there is a set $B \subset G$ with $|B| < k - 1$ such that $G - B$ has a d-scattered set of size m.

This is established by starting with a large set S in G so that we can repeatedly find a large enough subset S' and expand the radius of neighbourhoods of elements in S' while keeping them disjoint. To do this we may have to delete some elements of G, but we do not delete more than $k - 1$ elements in total.

This involves an iterated Ramsey argument.

Conclusion

The class of *all* finite structures is not well-behaved in a model-theoretic sense.

Putting additional structural restrictions allows us to recover some interesting model theory.

The restrictions often coincide with those giving interesting algorithmic properties.

Besides *Preservation* Theorems, many other properties remain to be explored.

In the absence of *Compactness*, the proof methods are varied and often highly combinatorial, though *Locality* plays an important role.