Advanced Universal Algebra and the CSP

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Outline

1. Constraint Languages and Algebra
2. Tractable Algebras
3. Tame Congruence Theory
4. Some special operations
5. Finite Width
6. Algebras with few subpowers
7. Conclusion

The constraint language determined by an algebra

Definition

Let $\mathbf{A} = \langle A, F \rangle$ be a finite algebra.

- The constraint language determined by $\mathbf{A}$ is the set of finitary relations over A that are invariant under the operations in F and is denoted by $\mathrm{Inv}(\mathbf{A})$.
- $\mathrm{CSP}(\mathbf{A})$ denotes $\mathrm{CSP}(\mathrm{Inv}(\mathbf{A}))$.
- We call $\mathbf{A}$ tractable (NP-complete) if the constraint language $\mathrm{Inv}(\mathbf{A})$ is.

Note

$\mathrm{Inv}(\mathbf{A})$ is equal to the set of all subuniverses of finite cartesian powers of $\mathbf{A}$.

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The dichotomy conjecture

Dichotomy Conjecture (Feder, Vardi):

Every constraint language is either tractable or NP-complete.

Definition

Let $\mathbf{A} = \langle A, F \rangle$ be a finite algebra.

- An operation $f(x_1, \ldots, x_n)$ on A is idempotent if $f(a, a, \ldots, a) = a$ for all $a \in A$.
- $\mathbf{A}$ is an idempotent algebra if each operation $f \in F$ is idempotent.

Theorem (Bulatov, Jeavons, Krokhin)

To prove the Dichotomy Conjecture, it suffices to verify it for the constraint languages $\mathrm{Inv}(\mathbf{A})$ *for* $\mathbf{A}$ *a finite idempotent algebra.*

The tractability conjecture

Conjecture (Bulatov, Jeavons, Krokhin)

A finite idempotent algebra $\mathbf{A}$ *is tractable if and only if* $\mathrm{var}(\mathbf{A})$ *omits the unary type. It is NP-complete otherwise.*

Note

Recall that $\mathrm{var}(\mathbf{A})$ denotes the variety generated by $\mathbf{A}$.

Questions

- What does it mean for an algebra or variety of algebras to omit the unary (or some other) type?
- How can one recognize when this occurs?

Tame Congruence Theory

Tame Congruence Theory

Hobby and McKenzie have developed a notion of neighbourhood, or minimal set of a finite algebra. They show that the behaviour of minimal sets is limited to one of the following five types:

1. Unary
2. Affine
3. 2-element Boolean algebra
4. 2-element Lattice
5. 2-element Semi-lattice

Definition

- We say that a finite algebra **A** omits a particular type if no neighbourhoods of that type occur in **A**.
- A variety $\mathcal{V}$ omits a particular type if each finite member of it does.

Localization

Definition

Let $\mathbf{A}$ be an algebra.

- A polynomial of $\mathbf{A}$ is an operation $p(x_1, \ldots, x_n)$ of the form $t(x_1, \ldots, x_n, a_1, \ldots, a_m)$ for some term t and elements a_i of $\mathbf{A}$.
- A unary polynomial $p(x)$ is idempotent if $p(p(x)) = p(x)$ for all $x \in A$, i.e., $p(x) = x$ for x in the range of p.

Definition

Let $p(x)$ be a unary idempotent polynomial of $\mathbf{A}$ and let $U = p(x)$. The localization of $\mathbf{A}$ to U is the algebra $\langle U, \mathcal{F} \rangle$ with universe U and

$$\mathcal{F} = \{p(t(\bar{x}))|_U \ : \ t(\bar{x}) \text{ a polynomial of } \mathbf{A}\}$$

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Neighbourhoods

Definition

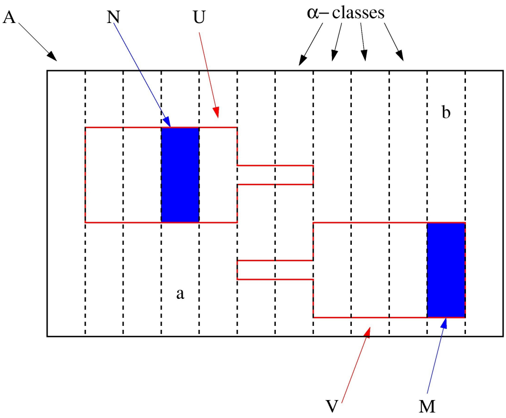
Let **A** be a finite algebra and α a minimal congruence of **A**.

- An α-minimal set of **A** is a subset U of A such that
  - $U = p(A)$ for some idempotent polynomial $p(x)$ of **A** that is not constant on the α-classes, and
  - U is minimal with this property.
- An α-neighbourhood of **A** is a subset N of A such that
  - $N = U \cap (a/\alpha)$ for some α-minimal set U and α-class (a/α), and
  - $|N| > 1$.

Facts

- *The localizations of all α-minimal sets and all neighbourhoods are isomorphic.*
- *The algebraic structure of each α-neighbourhood is of one of the five types mentioned earlier.*

Neighbourhoods



Legend

- A is partitioned by the α-classes.
- U and V are α-minimal sets.
- $N = U \cap (a/\alpha)$ and $M = V \cap (b/\alpha)$ are α-neighbourhoods.

Definition

The type of α is equal to the type of any one of the α-neighbourhoods.

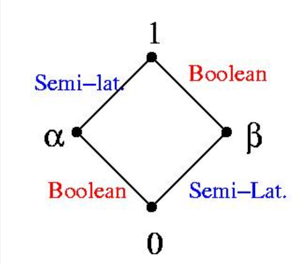
An Example

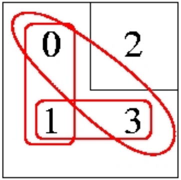
An algebra

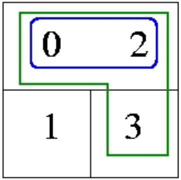
Let **A** be the algebra on $\{0, 1, 2, 3\}$ and with operation:

$\cdot$	0	1	2	3
0	0	0	0	3
1	0	1	0	1
2	0	0	2	3
3	3	1	3	3

Its labelled congruence lattice



$\alpha =$ 

$\beta =$ 

Unanimity Operations

Definition

Let A be a set and $t(x_1, \ldots, x_n)$ an operation on A with $n > 2$.

- t is a weak near-unanimity operation if it is idempotent and satisfies the equations:

$$t(y, x, x, \ldots, x) = t(x, y, x, \ldots, x) = \cdots = t(x, x, \ldots, x, y)$$

- t is a near-unanimity operation if it satisfies:

$$t(y, x, x, \ldots, x) = t(x, y, x, \ldots, x) = \cdots = t(x, x, \ldots, x, y) = x$$

Theorem (Cohen, Cooper, Jeavons)

If **A** *has a near-unanimity term operation then it is tractable.*

Examples

Examples

- Let $\mathbf{L} = \langle L, \vee, \wedge \rangle$ be a lattice. Then the operation

$$m(x, y, z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$$

  is a near unanimity operation.
- The operation $x_1 + x_2 + \cdots + x_{n+1}$ is a weak near-unanimity operation for the group of integers, modulo n.
- $x \cdot (y \cdot z)$ is a weak near-unanimity operation for the 4 element example.

Omitting the unary type

Theorem (Hobby-McKenzie, Maroti-McKenzie)

Let $\mathbf{A}$ *be a finite algebra. The following are equivalent:*

- $\mathrm{var}(\mathbf{A})$ *omits the unary type*
- $\mathbf{A}$ *has a Taylor term operation*
- *the congruence lattices of algebras in* $\mathrm{var}(\mathbf{A})$ *satisfy a particular (complicated) identity*
- $\mathbf{A}$ *has a weak near-unanimity term operation.*

Theorem (Bulatov, Jeavons)

If $\mathbf{A}$ *is idempotent then the above are equivalent to:*

- *No subalgebra of* $\mathbf{A}$ *admits the unary type.*

A new formulation of the Tractability Conjecture

Conjecture

Let **A** *be a finite idempotent algebra. Then* **A** *is tractable if and only if* **A** *has a weak near-unanimity term operation.* **A** *is NP-complete otherwise.*

Fact (Bulatov, Jeavons)

If this conjecture is correct then

- *there is a polynomial time algorithm to test for the tractability of* **A***.*
- *the problem of determining if a finite constraint language* Γ *is tractable is NP-complete.*

Challenge

Determine how the presence of a weak near-unanimity term operation for **A** implies that the subuniverses of cartesian powers of **A** are well behaved.

Tractability via local consistency

Definition

A finite algebra $\mathbf{A}$ (or constraint language Γ) has width k if the instances of CSP($\mathbf{A}$) (or CSP(Γ)) that have solutions are precisely those that are locally k-consistent (in some sense). It has finite width if it has width k for some k.

Fact

If Γ has finite width then it is globally tractable.

Note

- There are several closely related notions of width in the literature.
- The most prominent is one that is equivalent to a certain kind of definability within Datalog.

The finite width conjecture

Conjecture (Larose, Zadori)

Let **A** *be a finite idempotent algebra.* **A** *has finite width if and only if* var(**A**) *omits the unary and affine types.*

Notes

- Larose and Zadori have established one direction of this conjecture, namely that if **A** has finite width then var(**A**) omits the unary and affine types.
- Bulatov has proposed a similar conjecture but which is expressed using a different sort of local structure.

Omitting the unary and affine types

Theorem (Hobby, McKenzie)

Let **A** *be a finite algebra. The following are equivalent:*

- var(**A**) *omits the unary and affine types*
- *the congruence lattices of the algebras in* var(**A**) *are meet semi-distributive, i.e., satisfy the implication: if* $x \wedge y = x \wedge z$ *then* $x \wedge y = x \wedge (y \vee z)$,
- *[Kearnes, Szendrei]* **A** *has terms* $d_i(x, y, z)$, $e_i(x, y, z)$ *for* $1 \leq i \leq n$ *that satisfy the equations :*
  (i) $d_1(x, x, z) = x$, $d_n(x, x, z) = z$;
  (ii) $d_i(x, z, x) = d_{i+1}(x, z, x)$ *and* $e_i(x, z, x) = e_{i+1}(x, z, x)$, *if* i *is odd;*
  (iii) $d_i(x, x, z) = d_{i+1}(x, x, z)$, *if* i *is even,* $e_i(x, x, z) = e_{i+1}(x, x, z)$, *if* i *is odd; and*
  (iv) $d_i(x, z, z) = e_i(x, z, z)$ *for all* i.

A cleaner characterization

Theorem (McKenzie)

Let $\mathbf{A}$ *be a finite algebra. Then* $\mathrm{var}(\mathbf{A})$ *omits the unary and affine types if and only if for some* $N > 0$, $\mathbf{A}$ *has a* k*-variable weak near-unanimity term operation for all* $k > N$.

Notes

- McKenzie's proof employs the meet semi-distributivity of the congruence lattices of members of $\mathrm{var}(\mathbf{A})$.
- Kiss-Valeriote show that finite width implies the existence of weak near-unanimity term operations of almost all arities.
- The theorem provides strong support for the finite width conjecture.

Examples

Examples

A finite algebra with one of the following operations generates a variety that omits the unary and affine types (and has finite width and so is tractable):

- A semi-lattice operation $x \wedge y$, i.e., an operation that satisfies: $x \wedge x = x$, $x \wedge y = y \wedge x$, and $x \wedge (y \wedge z) = (x \wedge y) \wedge z$.
- A near-unanimity operation.

Note

For any $n > 2$ the term $x_1 \cdot (x_2 \cdot (x_3 \cdots (x_{n-1} \cdot x_n) \cdots)$ is a weak near unanimity operation for our 4 element example. Bulatov and Jeavons have shown that this algebra has finite width and hence is tractable.

The finite width conjecture, revisited

Note

Using McKenzie's result, the finite width conjecture can be restated as follows:

Conjecture (Finite Width Conjecture)

Let **A** *be a finite idempotent algebra. Then* **A** *has finite width if and only if for some* $N > 0$, **A** *has a* k*-variable weak near-unanimity term operation for all* $k > N$.

Facts

If the conjecture is true, it follows that

- *[Larose, Valeriote] there is a polynomial time algorithm to determine if a finite idempotent algebra has finite width.*
- *[Bulatov] the problem of determining if a finite constraint language has finite width is NP-complete.*

Sketch of Proof

Proof of one direction of McKenzie's result

- Suppose that $\mathbf{A}$ is a finite idempotent algebra such that for some N, $\mathbf{A}$ has k-ary weak near unanimity terms for all $k > N$.
- Show that $\text{var}(\mathbf{A})$ must omit the affine and unary types.
- If not, let $\mathbf{B} \in \text{var}(\mathbf{A})$ and α a minimal congruence of $\mathbf{B}$ of affine type.
- Let N be an α-neighbourhood of $\mathbf{B}$.
- Then the localization to N is a vector space, and
- it has a k-ary weak near unanimity term for each $k > N$.
- This is impossible, since vector spaces do not have this property.

Tractability via small generating sets

Note

The tractability results for

- (Feder, Vardi) groups,
- (Cohen, Cooper, Jeavons) near unanimity operations,
- (Bulatov) Mal'cev operations, and
- (Dalmau) generalized majority-minority (gmm) operations

can all be understood using the following algebraic property:

Definition

A finite algebra $\mathbf{A}$ has few subpowers if there is some polynomial $p(n)$ such that for each $n > 0$,

$$s_{\mathbf{A}}(n) = \log_2 |\{B : B \text{ is a subuniverse of } \mathbf{A}^n\}| \leq p(n).$$

Algebras with few subpowers

Note

Algebras with few subpowers have been studied in various guises by Bulatov, Chen, and Dalmau. Chen and Dalmau both conjectured that if **A** is a finite algebra having few subpowers then **A** is tractable.

Theorem (Idziak, Markovic, McKenzie, Valeriote, Willard)

If **A** *is a finite algebra having few subpowers then* **A** *is globally tractable.*

Note

The proof modifies Dalmau's gmm algorithm and uses the fact that if an algebra has few subpowers then all of its subpowers have small generating sets.

A characterization of algebras with few subpowers

Definition

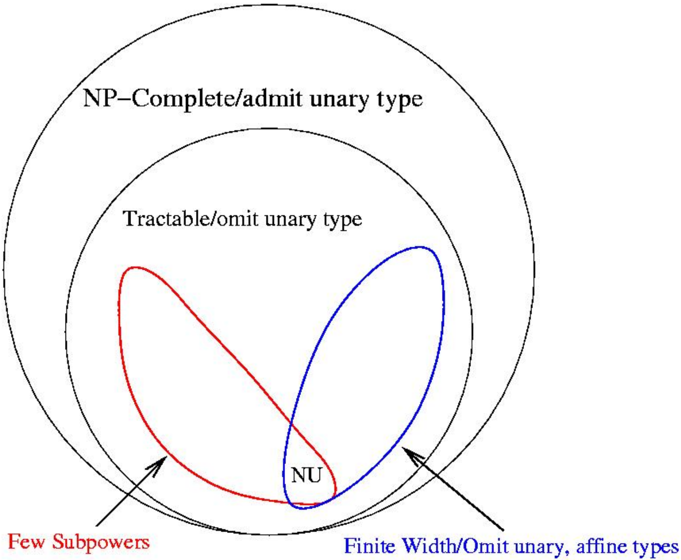
A k-edge operation on a set A is a $k+1$-variable operation t that satisfies the equations:

$$\begin{aligned}
t(x, x, y, y, y, \ldots, y, y) &= y \\
t(x, y, x, y, y, \ldots, y, y) &= y \\
t(y, y, y, x, y, \ldots, y, y) &= y \\
t(y, y, y, y, x, \ldots, y, y) &= y \\
&\vdots \\
t(y, y, y, y, y, \ldots, y, x) &= y.
\end{aligned}$$

Theorem (Berman, Idziak, Markovic, McKenzie, Valeriote, Willard)

Let $\mathbf{A}$ *be a finite algebra. Then* $\mathbf{A}$ *has few subpowers if and only if it has a* k*-edge term operation for some* $k > 1$.

Conclusion: The Big Picture



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