Fixed-parameter tractable constraint satisfaction

Marco Samer
TU Vienna, Austria

Stefan Szeider
U Durham, UK

Dagstuhl-Seminar 06401, Complexity of Constraints, October 2006

Talk Outline

- Refined view at hierarchies of tractable CSP (like bounded treewidth)
- Trade-off: Generality vs Performance
- Framework of Parameterized Complexity
- Continue line of research [Gottlob, Scarcello, Sideri 02]
- New hardness results, new tractability results
- Via "domination" complete classification for a large number of combinations of parameters
- Hardness results apply to general width parameters (like hypertree width) and to Boolean CSP

Tractable CSPs

- Tractable classes of CSP instances
- Hierarchies of tractable classes:

$$\mathcal{C}_1 \subset \mathcal{C}_2 \subset \cdots \subset \mathcal{C}_k \subset \ldots$$

- Hierarchies based on "width parameters" like *treewidth* or *hypertree width*

$$\mathcal{C}_k = \{\, I : \mathsf{width}(I) \leq k \,\}$$

i.e., restrictions on the left hand side.

Generality vs Performance

$$\begin{array}{ccccccccc} \mathcal{C}_1 & \subset & \mathcal{C}_2 & \subset \cdots \subset & \mathcal{C}_k & \subset \dots \\ O(n) & & O(n^2) & \dots & O(n^k) & \dots \\ O(c_1 n^\alpha) & & O(c_2 n^\alpha) & \dots & O(c_k n^\alpha) & \dots \end{array}$$

Cost of more generality:

- non-uniform polynomial time: pay with polynomial of higher order.
- uniform polynomial time: we pay only by a larger constant factor.

Research Question

Which hierarchies/parameterizations allow uniform polytime solution and which don't?

... and what theoretical framework is suitable for this investigation?

Parameterized Complexity

- "Two dimensional complexity theory"
- Initiated by Downey and Fellows in late 1980s
- Now significant branch in algorithm design with 100s of research papers (two new monographs [Flum & Grohe 06], [Niedermeier 06])

fixed-parameter tractable (FPT)	W[1]-hard
solvable in uniform polytime	very unlikely to be uniform polytime
rich toolkit for developing FPT algorithms	evidence via completeness theory.

- Distinction has been shown robust and well related to practicability.

Parameterized CSP

- *CSP parameter*: a computable function $p : \mathbf{CSP} \rightarrow \{1, 2, \ldots\}$.
- parameterized problem: **CSP**(p)
  - Instance: CSP instance I, integer k with $p(I) \leq k$.
  - Parameter: k.
  - Question: is I satisfiable?
- "promise problem"
- Also combination of parameters, **CSP**$(p_1, \ldots, p_r)$.

Basic CSP parameters

- **vars**: number of variables
- **dom**: size of domain
- **arity**: max arity of constraints
- **ovl**: max overlap of constraint scopes
- Observations:
  - **CSP**(**dom**) is not FPT.
    (3-colorability)
  - **CSP**(**vars**) is W[1]-hard
    [Papadimitriou & Yannakakis 99]
  - **CSP**(**dom**, **vars**) is FPT
    (check $\mathbf{dom}^{\mathbf{vars}}$ assignments)

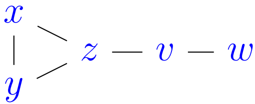
W[1]-hardness of **CSP**(**vars**) [P&Y 99]

- Parameterized reduction from *k-CLIQUE:*
- Given a graph $G = (V, E)$
- Construct CSP instance:
  - k variables $x_1, \ldots, x_k$.
  - domains: $D(x_i) = V$.
  - $\binom{k}{2}$ binary constraints $((x_i, x_j), E)$.

treewidth parameters

- **tw**(I) treewidth of *primal graph*

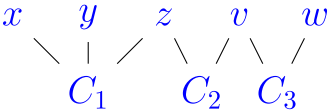
  Example: C_1 on x, y, z; C_2 on z, v; C_3 on v, w.



- **tw**$^d(I)$ treewidth of *dual graph*

$$C_1 - C_2 - C_3$$

- **tw**$^*(I)$ treewidth of *incidence graph* (most general)



Classification Theorem

Our new results allow to classify all combinations of the aforementioned parameters,

$$S \subseteq \{\textbf{tw}, \textbf{tw}^d, \textbf{tw}^*, \textbf{dom}, \textbf{arity}, \textbf{ovl}\}$$

whether $\textbf{CSP}(S)$ is FPT (uniformly polytime) or not.

... $2^6 = 64$ subsets to consider ...

Domination

- If **arity** is bounded, **ovl** gets automatically bounded.
- So **ovl** is the more general parameter. We say that **ovl** *dominates* **arity**.
- First consequence: not all combinations are interesting:
  E.g.: **CSP**$(\ldots,$ **ovl**, **arity**$)$ is already covered by **CSP**$(\ldots,$ **arity**$)$.
- Second consequence: If X dominates Y then
  - **CSP**(X) is FPT $\Rightarrow$ **CSP**(Y) is FPT
  - **CSP**(Y) is not FPT $\Rightarrow$ **CSP**(X) is not FPT.

Domination of width parameters

- **tw*** dominates **tw** [Kolaitis & Vardi 00].
- **tw*** dominates **tw**^{d} (by symmetric argument).
- **arity** dominates **tw** (large arity gives large cliques).
- **tw** dominates **vars** (trivial).
- 12 non-redundant subsets remain to consider, partially ordered by dominance:

The Domination Lattice

$$
\begin{tikzcd}[column sep=tiny]
 & & & \{\mathbf{tw}^*\} \arrow[dll] \arrow[d] \arrow[drr] & & & \\
 & \{\mathbf{tw}^d\} \arrow[dl] \arrow[dr] & & \{\mathbf{tw}^*, \mathbf{dom}\} \arrow[dlll] \arrow[dr] & & \{\mathbf{tw}^*, \mathbf{ovl}\} \arrow[dlll] \arrow[dl] \arrow[dr] & \\
\{\mathbf{tw}^d, \mathbf{dom}\} \arrow[dr] & & \{\mathbf{tw}^d, \mathbf{ovl}\} \arrow[dl] \arrow[dr] & & \{\mathbf{tw}^*, \mathbf{dom}, \mathbf{ovl}\} \arrow[dlll] \arrow[dr] & & \{\mathbf{tw}\} \arrow[dlll] \arrow[dl] \\
 & \{\mathbf{tw}^d, \mathbf{dom}, \mathbf{ovl}\} \arrow[drr] & & \{\mathbf{tw}^d, \mathbf{arity}\} \arrow[d] & & \{\mathbf{tw}, \mathbf{dom}\} \arrow[dll] & \\
 & & & \{\mathbf{tw}^d, \mathbf{dom}, \mathbf{arity}\} & & &
\end{tikzcd}
$$

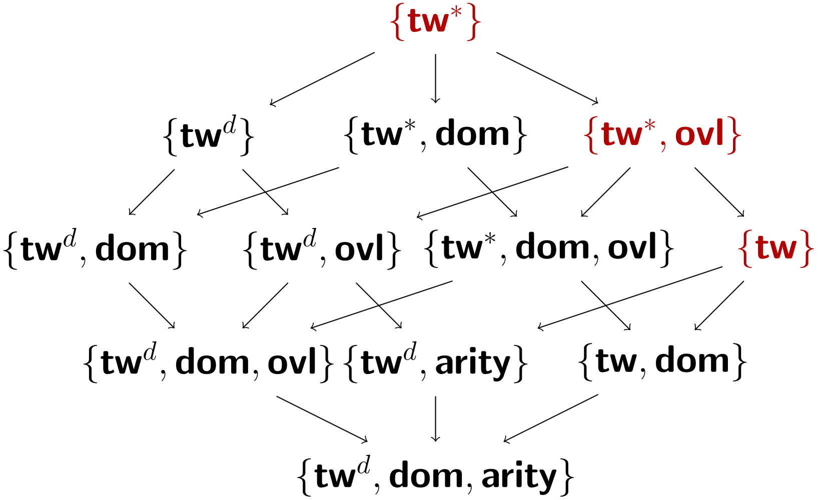
W[1]-hardness via [P&Y 99]

$$
\begin{tikzcd}[column sep=tiny]
 & & & \{\mathbf{tw}^*\} \arrow[dll] \arrow[d] \arrow[drr] & & & \\
 & \{\mathbf{tw}^d\} \arrow[dl] \arrow[dr] & & \{\mathbf{tw}^*, \mathbf{dom}\} \arrow[dlll] \arrow[dr] & & \{\mathbf{tw}^*, \mathbf{ovl}\} \arrow[dlll] \arrow[dl] \arrow[dr] & \\
\{\mathbf{tw}^d, \mathbf{dom}\} \arrow[dr] & & \{\mathbf{tw}^d, \mathbf{ovl}\} \arrow[dl] \arrow[dr] & & \{\mathbf{tw}^*, \mathbf{dom}, \mathbf{ovl}\} \arrow[dlll] \arrow[dr] & & \{\mathbf{tw}\} \arrow[dlll] \arrow[dl] \\
 & \{\mathbf{tw}^d, \mathbf{dom}, \mathbf{ovl}\} \arrow[drr] & & \{\mathbf{tw}^d, \mathbf{arity}\} \arrow[d] & & \{\mathbf{tw}, \mathbf{dom}\} \arrow[dll] & \\
 & & & \{\mathbf{tw}^d, \mathbf{dom}, \mathbf{arity}\} & & &
\end{tikzcd}
$$

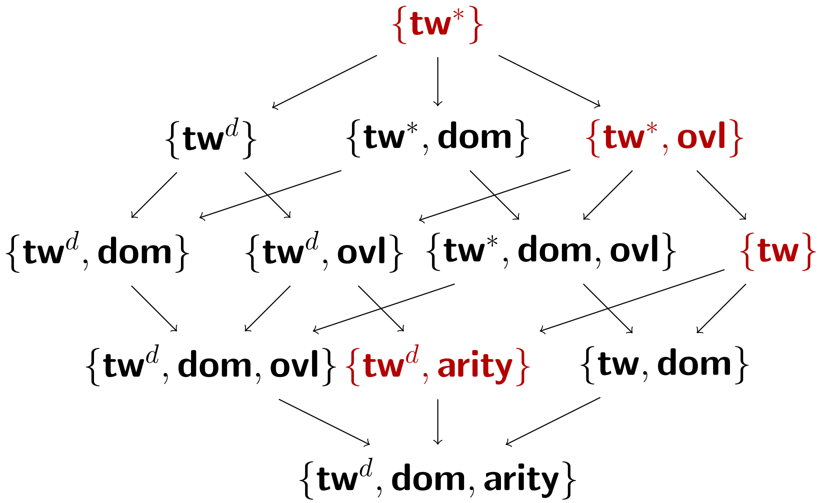
W[1]-hardness via [P&Y 99]

$$
\begin{tikzcd}
 & & & \{\mathbf{tw}^*\} \arrow[dll] \arrow[d] \arrow[drr] & & & \\
 & \{\mathbf{tw}^d\} \arrow[dl] \arrow[dr] & & \{\mathbf{tw}^*, \mathbf{dom}\} \arrow[dlll] \arrow[dr] & & \{\mathbf{tw}^*, \mathbf{ovl}\} \arrow[dlll] \arrow[dl] \arrow[dr] & \\
\{\mathbf{tw}^d, \mathbf{dom}\} \arrow[dr] & & \{\mathbf{tw}^d, \mathbf{ovl}\} \arrow[dl] \arrow[dr] & & \{\mathbf{tw}^*, \mathbf{dom}, \mathbf{ovl}\} \arrow[dlll] \arrow[dr] & & \{\mathbf{tw}\} \arrow[dlll] \arrow[dl] \\
 & \{\mathbf{tw}^d, \mathbf{dom}, \mathbf{ovl}\} \arrow[drr] & & \{\mathbf{tw}^d, \mathbf{arity}\} \arrow[d] & & \{\mathbf{tw}, \mathbf{dom}\} \arrow[dll] & \\
 & & & \{\mathbf{tw}^d, \mathbf{dom}, \mathbf{arity}\} & & &
\end{tikzcd}
$$

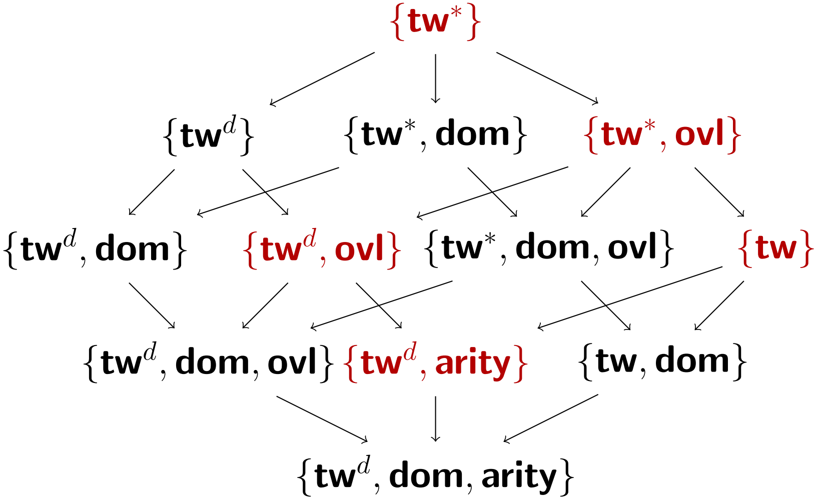
W[1]-hardness via [P&Y 99]



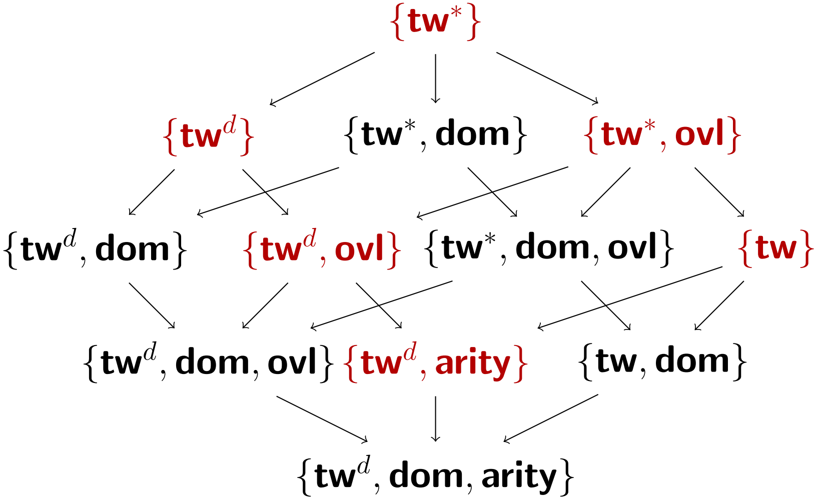
Same reduction works!



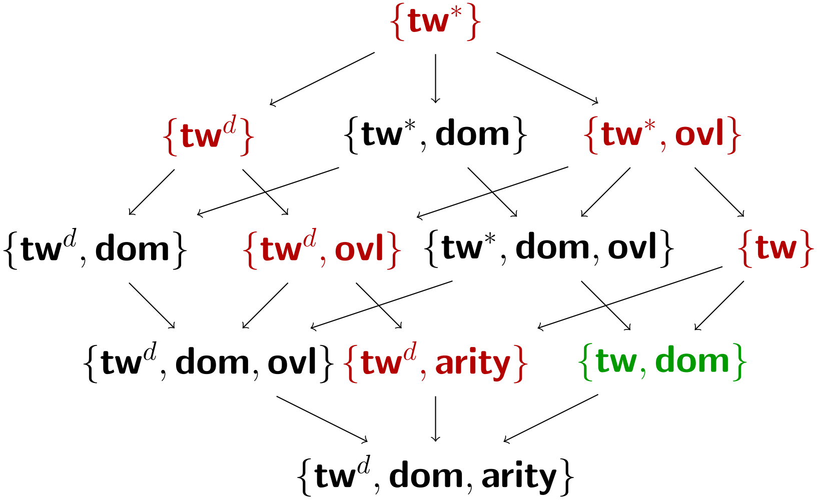
Same reduction works!



Same reduction works!



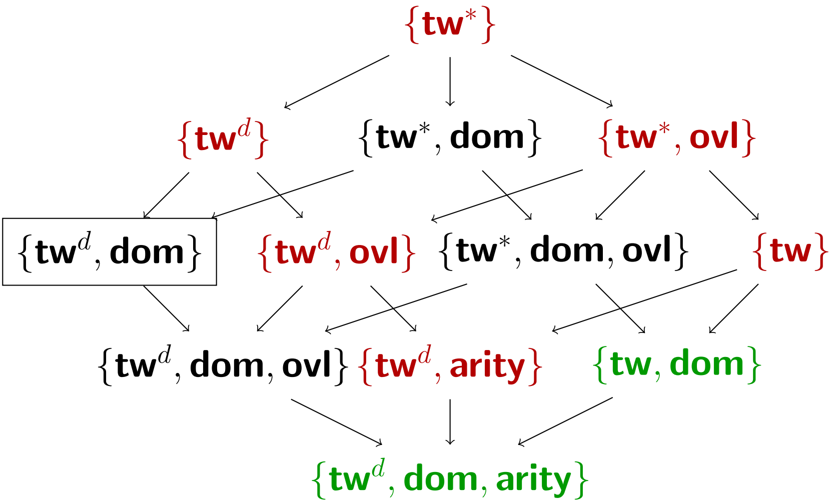
FPT by [Gottlob, Scarcello, Sideri 02]



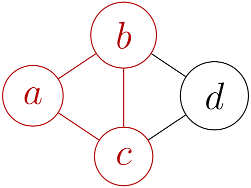
FPT by [Gottlob, Scarcello, Sideri 02]

$$\begin{tikzcd}[column sep=tiny]
 & & & \{\mathbf{tw}^*\} \arrow[dll] \arrow[d] \arrow[drr] & & & \\
 & \{\mathbf{tw}^d\} \arrow[dl] \arrow[dr] & & \{\mathbf{tw}^*, \mathbf{dom}\} \arrow[dlll] \arrow[dr] & & \{\mathbf{tw}^*, \mathbf{ovl}\} \arrow[dlll] \arrow[dl] \arrow[dr] & \\
\{\mathbf{tw}^d, \mathbf{dom}\} \arrow[dr] & & \{\mathbf{tw}^d, \mathbf{ovl}\} \arrow[dl] \arrow[dr] & & \{\mathbf{tw}^*, \mathbf{dom}, \mathbf{ovl}\} \arrow[dlll] \arrow[dr] & & \{\mathbf{tw}\} \arrow[dlll] \arrow[dl] \\
 & \{\mathbf{tw}^d, \mathbf{dom}, \mathbf{ovl}\} \arrow[drr] & & \{\mathbf{tw}^d, \mathbf{arity}\} \arrow[d] & & \{\mathbf{tw}, \mathbf{dom}\} \arrow[dll] & \\
 & & & \{\mathbf{tw}^d, \mathbf{dom}, \mathbf{arity}\} & & &
\end{tikzcd}$$

Complexity of **CSP**($\mathbf{tw}^d$, **dom**)



Reduction from k-CLIQUE



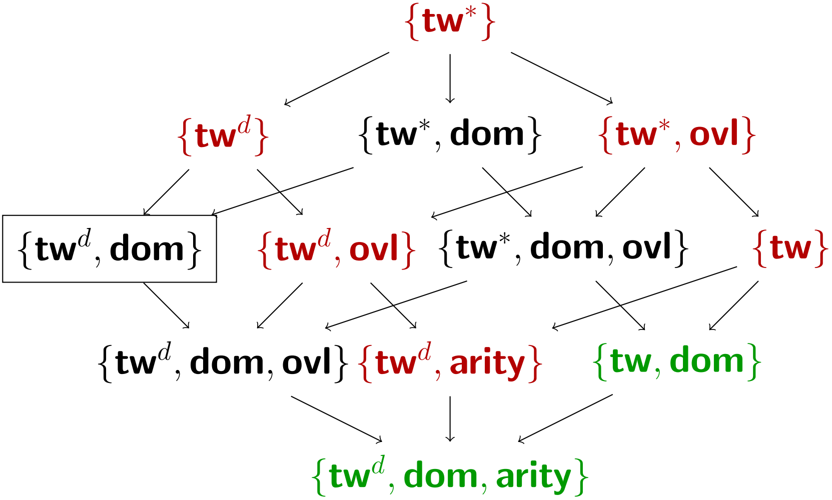
Solution
$a_1 = 1,\ b_2 = 1,\ c_3 = 1$,
(all other variables $= 0$)
corresponds to 3-clique
a, b, c.
treewidth of dual graph is bounded

a_1	b_1	c_1	d_1	a_2	b_2	c_2	d_2
1	0	0	0	0	1	0	0
1	0	0	0	0	0	1	0
0	1	0	0	0	0	1	0
0	1	0	0	0	0	0	1
0	0	1	0	0	0	0	1

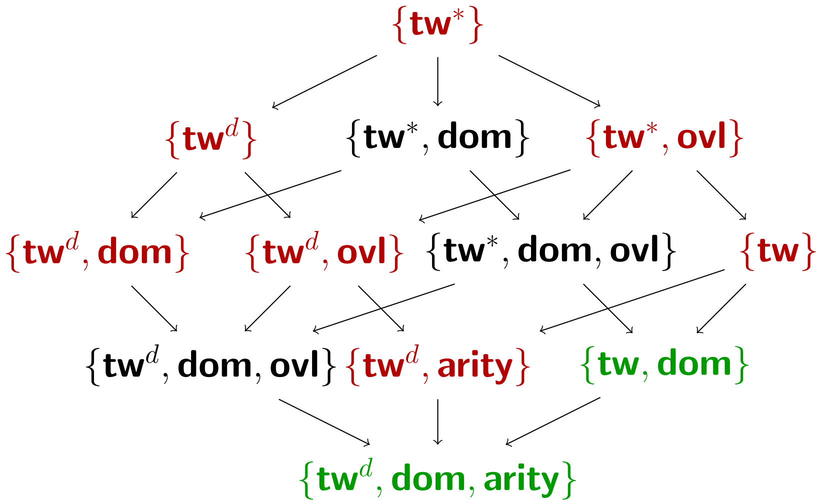
a_1	b_1	c_1	d_1	a_3	b_3	c_3	d_3
1	0	0	0	0	1	0	0
1	0	0	0	0	0	1	0
0	1	0	0	0	0	1	0
0	1	0	0	0	0	0	1
0	0	1	0	0	0	0	1

a_2	b_2	c_2	d_2	a_3	b_3	c_3	d_3
1	0	0	0	0	1	0	0
1	0	0	0	0	0	1	0
0	1	0	0	0	0	1	0
0	1	0	0	0	0	0	1
0	0	1	0	0	0	0	1

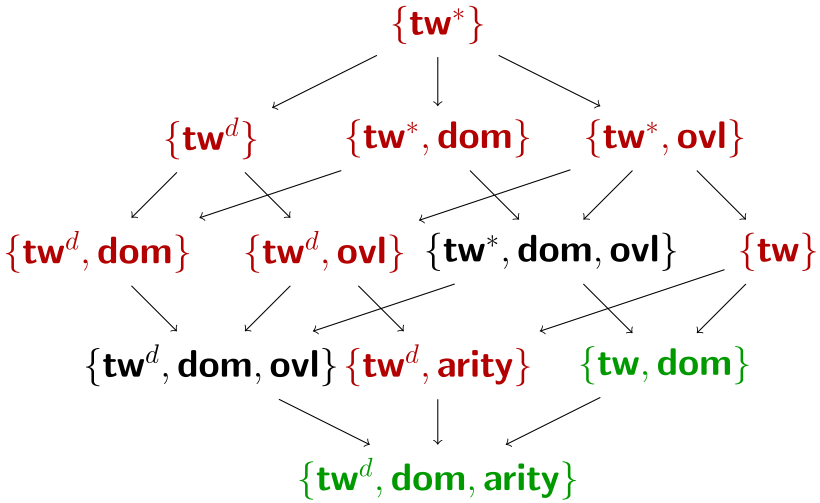
Updated Lattice



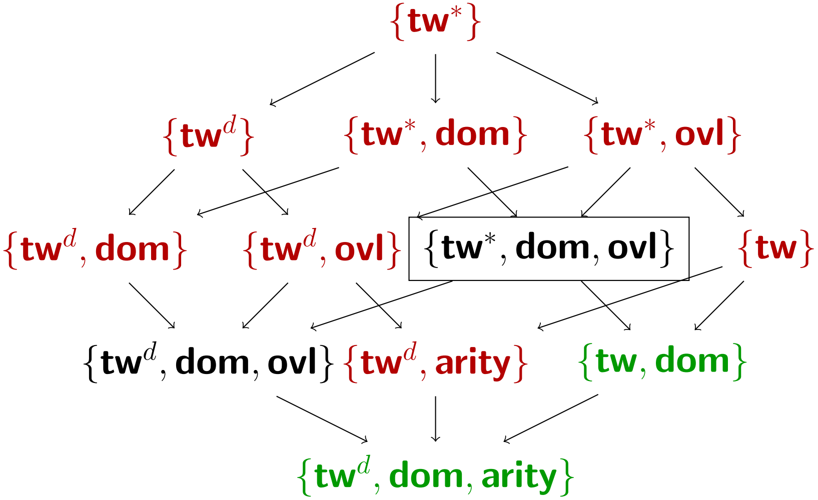
Updated Lattice



Updated Lattice



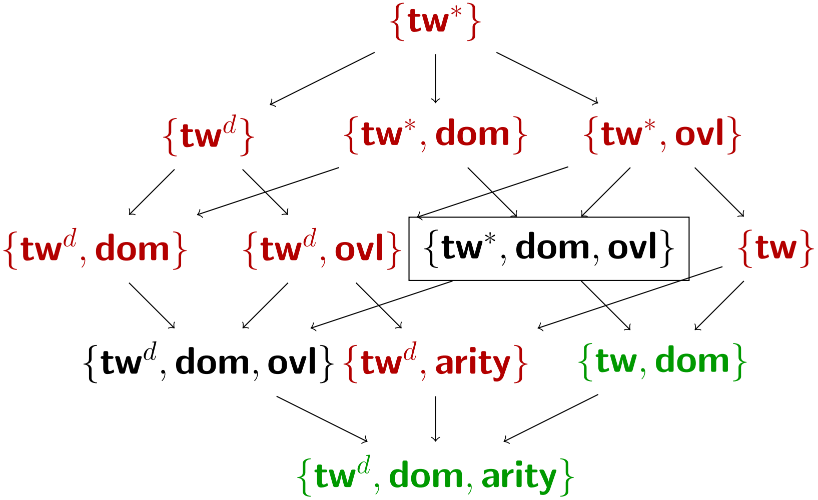
Complexity of $\mathbf{CSP}(\mathbf{tw}^*, \mathbf{dom}, \mathbf{ovl})$?



New FPT result

- *Theorem:* $\mathbf{CSP}(\mathbf{tw}^*, \mathbf{dom}, \mathbf{ovl})$ is FPT.
- via *dynamic programming* algorithm on "nice" tree-decompositions
  partial solutions are collected and in a bottom-up traversal of tree-decomposition.
- Technically complicated since we need to deal with "partial constraints."

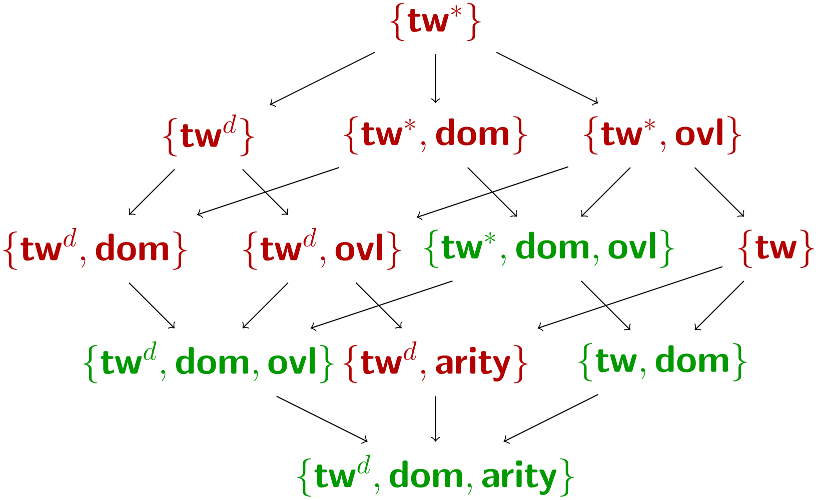
Updated lattice



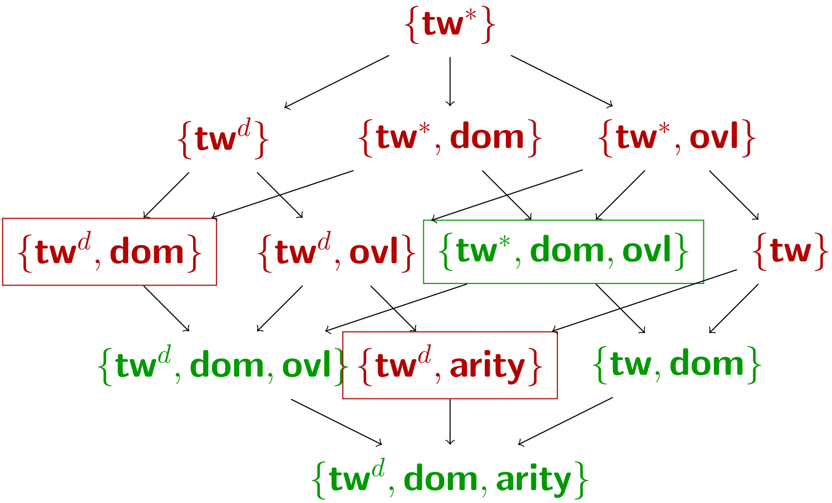
Updated lattice

$$
\begin{tikzcd}[column sep=tiny]
 & & & \{\mathbf{tw}^*\} \arrow[dll] \arrow[d] \arrow[drr] & & & \\
 & \{\mathbf{tw}^d\} \arrow[dl] \arrow[dr] & & \{\mathbf{tw}^*, \mathbf{dom}\} \arrow[dlll] \arrow[dr] & & \{\mathbf{tw}^*, \mathbf{ovl}\} \arrow[dlll] \arrow[dl] \arrow[dr] & \\
\{\mathbf{tw}^d, \mathbf{dom}\} \arrow[dr] & & \{\mathbf{tw}^d, \mathbf{ovl}\} \arrow[dl] \arrow[dr] & & \{\mathbf{tw}^*, \mathbf{dom}, \mathbf{ovl}\} \arrow[dlll] \arrow[dr] & & \{\mathbf{tw}\} \arrow[dlll] \arrow[dl] \\
 & \{\mathbf{tw}^d, \mathbf{dom}, \mathbf{ovl}\} \arrow[drr] & & \{\mathbf{tw}^d, \mathbf{arity}\} \arrow[d] & & \{\mathbf{tw}, \mathbf{dom}\} \arrow[dll] & \\
 & & & \{\mathbf{tw}^d, \mathbf{dom}, \mathbf{arity}\} & & &
\end{tikzcd}
$$

Updated lattice



Border cases



Stronger parameterizations

- More general parameters have been suggested (defined in terms of constraint hypergraphs), such as:
  - *(generalized) hypertree width* [Gottlob, Leone, Scarcello 01]
  - *spread-cut width* [Cohen, Jeavons, Gyssens 05]
  - *fractional hypertree width* [Grohe & Marx 06]
- It was open whether $\mathbf{CSP}(p, \mathbf{dom})$ is fixed-parameter tractable for the above parameters p.

Further hardness results

- All the above parameters p dominate $\mathbf{tw}^*$.
- Hence: $\mathbf{CSP}(p, \mathbf{dom})$ *is W[1]-hard.*
- Note 1:
  - Recognizing instances of bounded hypertree-width is W[2]-hard [Gottlob, Grohe, Musliu, Samer, Scarcello, WG'05]
  - However, there is a (non-uniform) polytime algorithm for CSP of bounded hypertree width that *avoids the recognition process* [Dalmau & Chen CP05].
- Note 2: hardness results hold for $\mathbf{CSP}_{\text{boole}}(p)$.

Boolean CSP vs SAT

- New hardness actually holds for Boolean CSP.
- i.e., $\mathbf{CSP}_{boole}(\mathbf{tw}^*)$ is W[1]-hard.
- In contrast to $\mathbf{SAT}(\mathbf{tw}^*)$ which is FPT.

Summary

- More refined view on hierarchies of tractable CSP
- *Generality vs performance* in terms of parameterized complexity
- New hardness results for general width parameters.
- New fixed-parameter algorithm for $\mathbf{CSP}(\mathbf{dom}, \mathbf{ovl}, \mathbf{tw}^*)$ that captures all previously known FPT cases
- Domination as tool to ease classification
- Classified all combinations of basic and treewidth parameters.

Further research

- Study parameterized complexity of further parameters.
- Parameters defined via *backdoor sets* (results for SAT).
- Any interesting parameters based on constraint languages?