Fixed-parameter tractable constraint satisfaction

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Talk Outline

- Refined view at hierarchies of tractable CSP (like bounded treewidth)
- Trade-off: Generality vs Performance
- Framework of Parameterized Complexity
- Continue line of research [Gottlob, Scarcello, Sideri 02]
- New hardness results, new tractability results
- Via “domination” complete classification for a large number of combinations of parameters
- Hardness results apply to general width parameters (like hypertree width) and to Boolean CSP
Tractable CSPs

- Tractable classes of CSP instances
- Hierarchies of tractable classes:

  \[ \mathcal{C}_1 \subset \mathcal{C}_2 \subset \cdots \subset \mathcal{C}_k \subset \ldots \]

- Hierarchies based on “width parameters” like treewidth or hypertree width

  \[ \mathcal{C}_k = \{ I : \text{width}(I) \leq k \} \]

  i.e., restrictions on the left hand side.
Generality vs Performance

\[ C_1 \subset C_2 \subset \cdots \subset C_k \subset \ldots \]

\[ O(n) \quad O(n^2) \quad \ldots \quad O(n^k) \quad \ldots \]

\[ O(c_1 n^\alpha) \quad O(c_2 n^\alpha) \quad \ldots \quad O(c_k n^\alpha) \quad \ldots \]

Cost of more generality:

- **non-uniform polynomial time**: pay with polynomial of higher order.
- **uniform polynomial time**: we pay only by a larger constant factor.

Fixed-parameter tractable constraint satisfaction
Research Question

Which hierarchies/parameterizations allow uniform polytime solution and which don’t?

... and what theoretical framework is suitable for this investigation?
Parameterized Complexity

- “Two dimensional complexity theory”
- Initiated by Downey and Fellows in late 1980s
- Now significant branch in algorithm design with 100s of research papers (two new monographs [Flum & Grohe 06], [Niedermeier 06])

fixed-parameter tractable (FPT)
solvable in uniform polytime
rich toolkit for developing FPT algorithms

W[1]-hard
very unlikely to be uniform polytime
evidence via completeness theory.

- Distinction has been shown robust and well related to practicability.
Parameterized CSP

- **CSP parameter**: a computable function $p : \text{CSP} \rightarrow \{1, 2, \ldots \}$.
- **Parameterized problem**: $\text{CSP}(p)$
  - Instance: CSP instance $I$, integer $k$ with $p(I) \leq k$.
  - Parameter: $k$.
  - Question: is $I$ satisfiable?

- “promise problem”
- Also combination of parameters, $\text{CSP}(p_1, \ldots, p_r)$. 

Fixed-parameter tractable constraint satisfaction
Basic CSP parameters

- **vars**: number of variables
- **dom**: size of domain
- **arity**: max arity of constraints
- **ovl**: max overlap of constraint scopes

Observations:
- **CSP**(dom) is not FPT.
  (3-colorability)
- **CSP**(vars) is W[1]-hard
  [Papadimitriou & Yannakakis 99]
- **CSP**(dom, vars) is FPT
  (check dom^{vars} assignments)
Parameterized reduction from \( k\text{-CLIQUE} \):

Given a graph \( G = (V, E) \)

Construct CSP instance:

- \( k \) variables \( x_1, \ldots, x_k \).
- domains: \( D(x_i) = V \).
- \( \binom{k}{2} \) binary constraints \( ((x_i, x_j), E) \).
treewidth parameters

- **$\mathbf{tw}(I)$** treewidth of *primal graph*
  Example: $C_1$ on $x, y, z$; $C_2$ on $z, v$; $C_3$ on $v, w$.

- **$\mathbf{tw}^d(I)$** treewidth of *dual graph*

- **$\mathbf{tw}^*(I)$** treewidth of *incidence graph* (most general)

Fixed-parameter tractable constraint satisfaction
Our new results allow to classify all combinations of the aforementioned parameters,

\[ S \subseteq \{ \text{tw}, \text{tw}^d, \text{tw}^*, \text{dom}, \text{arity}, \text{ovl} \} \]

whether \( \text{CSP}(S) \) is FPT (uniformly polytime) or not.

... \( 2^6 = 64 \) subsets to consider ...
Domination

- If \textit{arity} is bounded, \textit{ovl} gets automatically bounded.
- So \textit{ovl} is the more general parameter. We say that \textit{ovl dominates arity}.
- First consequence: not all combinations are interesting:
  E.g.: $\text{CSP}(\ldots, \text{ovl}, \text{arity})$ is already covered by $\text{CSP}(\ldots, \text{arity})$.
- Second consequence: If $X$ dominates $Y$ then
  - $\text{CSP}(X)$ is FPT $\Rightarrow$ $\text{CSP}(Y)$ is FPT
  - $\text{CSP}(Y)$ is not FPT $\Rightarrow$ $\text{CSP}(X)$ is not FPT.
Domination of width parameters

- $tw^*$ dominates $tw$  
  [Kolaitis & Vardi 00].
- $tw^*$ dominates $tw^d$  
  (by symmetric argument).
- *arity* dominates $tw$  
  (large arity gives large cliques).
- $tw$ dominates $vars$  
  (trivial).
- 12 non-redundant subsets remain to consider, partially ordered by dominance:
The Domination Lattice

\{tw^*\} \\
\{tw^d\} \quad \{tw^*, dom\} \quad \{tw^*, ovl\}

\{tw^d, dom\} \quad \{tw^d, ovl\} \quad \{tw^*, dom, ovl\} \quad \{tw\}

\{tw^d, dom, ovl\} \quad \{tw^d, arity\} \quad \{tw, dom\}

\{tw^d, dom, arity\}

Fixed-parameter tractable constraint satisfaction
W[1]-hardness via [P&Y 99]
W[1]-hardness via \[ \text{[P&Y 99]} \]

\[
\{\text{tw}^*\} \\
\{\text{tw}^d\} \quad \{\text{tw}^*, \text{dom}\} \quad \{\text{tw}^*, \text{ovl}\} \\
\{\text{tw}^d, \text{dom}\} \quad \{\text{tw}^d, \text{ovl}\} \quad \{\text{tw}^*, \text{dom}, \text{ovl}\} \quad \{\text{tw}\} \\
\{\text{tw}^d, \text{dom}, \text{ovl}\} \quad \{\text{tw}^d, \text{arity}\} \quad \{\text{tw}, \text{dom}\} \\
\{\text{tw}^d, \text{dom}, \text{arity}\}
\]
W[1]-hardness via [P&Y 99]

Fixed-parameter tractable constraint satisfaction
Same reduction works!

\begin{itemize}
    \item \{tw^*\}
    \item \{tw^d\}
    \item \{tw^*, dom\}
    \item \{tw^*, ovlan\}
    \item \{tw^d, dom\}
    \item \{tw^d, ovlan\}
    \item \{tw^*, dom, ovlan\}
    \item \{tw^d, dom\}
    \item \{tw^d, dom, ovlan\}
    \item \{tw^d, dom, ovlan\}
    \item \{tw, dom\}
    \item \{tw, dom\}
    \item \{tw^d, dom, arity\}
    \item \{tw^d, dom, arity\}
\end{itemize}

Fixed-parameter tractable constraint satisfaction
Same reduction works!

Fixed-parameter tractable constraint satisfaction
Same reduction works!

\{\text{tw}^*\}

\{\text{tw}^d\}

\{\text{tw}^*, \text{dom}\}

\{\text{tw}^*, \text{ovl}\}

\{\text{tw}^d, \text{dom}\}

\{\text{tw}^d, \text{ovl}\}

\{\text{tw}^*, \text{dom}, \text{ovl}\}

\{\text{tw}\}

\{\text{tw}^d, \text{dom}, \text{ovl}\}

\{\text{tw}^d, \text{arity}\}

\{\text{tw}, \text{dom}\}

\{\text{tw}^d, \text{dom}, \text{arity}\}

Fixed-parameter tractable constraint satisfaction
FPT by [Gottlob, Scarcello, Sideri 02]
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Fixed-parameter tractable constraint satisfaction
Complexity of $\text{CSP}(\text{tw}^d, \text{dom})$
Reduction from $k$-CLIQUE

Solution

$a_1 = 1$, $b_2 = 1$, $c_3 = 1$, (all other variables = 0) corresponds to 3-clique $a, b, c$.

treewidth of dual graph is bounded
Fixed-parameter tractable constraint satisfaction
Fixed-parameter tractable constraint satisfaction
Updated Lattice

\{ \text{tw}^{*}\} \\
\{ \text{tw}^{d}\} \quad \{ \text{tw}^{*}, \text{dom}\} \quad \{ \text{tw}^{*}, \text{ovl}\} \\
\{ \text{tw}^{d}, \text{dom}\} \quad \{ \text{tw}^{d}, \text{ovl}\} \quad \{ \text{tw}^{*}, \text{dom}, \text{ovl}\} \quad \{ \text{tw}\} \\
\{ \text{tw}^{d}, \text{dom}, \text{ovl}\} \quad \{ \text{tw}^{d}, \text{arity}\} \quad \{ \text{tw}, \text{dom}\} \\
\{ \text{tw}^{d}, \text{dom}, \text{arity}\}
Complexity of \( \text{CSP}(\text{tw}^*, \text{dom}, \text{ovl})? \)

\[
\begin{align*}
\{\text{tw}^*\} \\
\{\text{tw}^d\} & \quad \{\text{tw}^*, \text{dom}\} & \{\text{tw}^*, \text{ovl}\} \\
\{\text{tw}^d, \text{dom}\} & \quad \{\text{tw}^d, \text{ovl}\} \\
\{\text{tw}^d, \text{dom}, \text{ovl}\} & \quad \{\text{tw}^d, \text{arity}\} & \{\text{tw}, \text{dom}\} \\
\{\text{tw}^d, \text{dom}, \text{arity}\} & \\
\{\text{tw}^d\}
\end{align*}
\]

Fixed-parameter tractable constraint satisfaction
New FPT result

- **Theorem:** $\text{CSP}(\text{tw}^*, \text{dom}, \text{ovl})$ is FPT.
- via *dynamic programming* algorithm on “nice” tree-decompositions. Partial solutions are collected and in a bottom-up traversal of tree-decomposition.
- Technically complicated since we need to deal with “partial constraints.”
Updated lattice

\{tw^*\} \quad \{tw^d\} \quad \{tw^*, \text{dom}\} \quad \{tw^*, \text{ovl}\}

\{tw^d, \text{dom}\} \quad \{tw^d, \text{ovl}\} \quad \{tw^*, \text{dom, ovl}\} \quad \{tw\}

\{tw^d, \text{dom, ovl}\} \quad \{tw^d, \text{arity}\} \quad \{tw, \text{dom}\}

\{tw^d, \text{dom, arity}\}

Fixed-parameter tractable constraint satisfaction
Updated lattice

\[
\begin{align*}
\{\text{tw}^*\} & \quad \{\text{tw}^d\} & \quad \{\text{tw}^*, \text{dom}\} & \quad \{\text{tw}^*, \text{ovl}\} \\
\{\text{tw}^d, \text{dom}\} & \quad \{\text{tw}^d, \text{ovl}\} & \quad \{\text{tw}^*, \text{dom}, \text{ovl}\} & \quad \{\text{tw}\} \\
\{\text{tw}^d, \text{dom}, \text{ovl}\} & \quad \{\text{tw}^d, \text{arity}\} & \quad \{\text{tw}, \text{dom}\} \\
\{\text{tw}^d, \text{dom}, \text{arity}\} & \\
\end{align*}
\]
Updated lattice

\{tw^*\} 

\{tw^d\} \quad \{tw^*, dom\} \quad \{tw^*, ovl\}

\{tw^d, dom\} \quad \{tw^d, ovl\} \quad \{tw^*, dom, ovl\} \quad \{tw\}

\{tw^d, dom, ovl\} \quad \{tw^d, arity\} \quad \{tw, dom\}

\{tw^d, dom, arity\}
Border cases

\{ \text{tw}^* \} \\
\{ \text{tw}^d \} \quad \{ \text{tw}^*, \text{dom} \} \quad \{ \text{tw}^*, \text{ovl} \} \\
\{ \text{tw}^d, \text{dom} \} \quad \{ \text{tw}^d, \text{ovl} \} \quad \{ \text{tw}^*, \text{dom}, \text{ovl} \} \quad \{ \text{tw} \} \\
\{ \text{tw}^d, \text{dom}, \text{ovl} \} \quad \{ \text{tw}^d, \text{arity} \} \quad \{ \text{tw}, \text{dom} \} \\
\{ \text{tw}^d, \text{dom}, \text{arity} \} \\

Fixed-parameter tractable constraint satisfaction
Stronger parameterizations

More general parameters have been suggested (defined in terms of constraint hypergraphs), such as:

- (generalized) hypertree width
  [Gottlob, Leone, Scarcello 01]
- spread-cut width
  [Cohen, Jeavons, Gyssens 05]
- fractional hypertree width
  [Grohe & Marx 06]

It was open whether \( \text{CSP}(p, \text{dom}) \) is fixed-parameter tractable for the above parameters \( p \).
Further hardness results

- All the above parameters $p$ dominate $\text{tw}^*$.
- Hence: $\text{CSP}(p, \text{dom})$ is $W[1]$-hard.
- Note 1:
  - However, there is a (non-uniform) polytime algorithm for CSP of bounded hypertree width that avoids the recognition process [Dalmau & Chen CP05].
- Note 2: hardness results hold for $\text{CSP}_{\text{boole}}(p)$. 

Fixed-parameter tractable constraint satisfaction
Boolean CSP vs SAT

- New hardness actually holds for Boolean CSP.
  - i.e., $\text{CSP}_{\text{boole}}(\text{tw}^*)$ is W[1]-hard.
  - In contrast to $\text{SAT}(\text{tw}^*)$ which is FPT.
Summary

- More refined view on hierarchies of tractable CSP
- *Generality vs performance* in terms of parameterized complexity
- New hardness results for general width parameters.
- New fixed-parameter algorithm for $\text{CSP}(\text{dom}, \text{ovl}, \text{tw}^*)$ that captures all previously known FPT cases
- Domination as tool to ease classification
- Classified all combinations of basic and treewidth parameters.
Further research

- Study parameterized complexity of further parameters.
- Parameters defined via *backdoor sets* (results for SAT).
- Any interesting parameters based on constraint languages?