Partial Clones and Constraint Satisfaction

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Co-Clones

$R \in \langle \Gamma \rangle$ iff there is some $\Gamma \cup \{=\}$-formula ϕ such that

$$R(x_1, \ldots, x_n) \equiv \exists y_1 \ldots y_\ell \; \phi(x_1, \ldots, x_n, y_1, \ldots, y_\ell)$$

- $R(x_1, \ldots, x_n)$ is satisfiability-equivalent to ϕ
- $\mathrm{CSP}(R) \leq_m^P \mathrm{CSP}(\Gamma)$
- Complexity $\mathrm{CSP}(\Gamma)$ depends only on co-clone.

No obvious reduction for some other problems

ENUM(Γ):

Input: Γ-formula ϕ

Task: Enumerate all solutions of ϕ

Boolean case:

Complexity depends only on $\langle\Gamma\rangle$. [Creignou, Hébrard 1997]

But no obvious reduction.

Equivalence

EQ(Γ):

Input: two Γ-formulas ϕ and ψ

Question: Is ϕ equivalent to ψ?

Boolean case:

Complexity of EQ(Γ) depends only on $\langle\Gamma\rangle$ [Böhler et al. 2002]

But no obvious reduction

Existence-Free Co-Clones

Define existence-free co-clone of Γ:

$R \in \langle \Gamma \rangle_{\nexists}$ iff there is some $\Gamma \cup \{=\}$-formula ϕ such that

$$R(x_1, \ldots, x_n) \equiv \phi(x_1, \ldots, x_n)$$

If $\langle \Gamma_1 \rangle_{\nexists} = \langle \Gamma_2 \rangle_{\nexists}$ then

- $\mathrm{EQ}(\Gamma_1) \equiv_m^P \mathrm{EQ}(\Gamma_2)$
- $\mathrm{ENUM}(\Gamma_1)$ and $\mathrm{ENUM}(\Gamma_2)$ have the same complexity

Existence-free co-clones are refinement of co-clones.

Question: What refinement of clones corresponds to existence-free co-clones?

Answer: Partial clones

Partial Clones

Let $E \subseteq D^n$. Then $f : E \to D$ is a partial function on D.

C is a strongly partial clone if

- C includes all projections
- C is closed under composition
- $f \in C$ implies that all restrictions of f are in C.

$[B]_{\text{par}}$ is the strongly partial clone generated by B.

Partial Polymorphisms

f is partial polymorphism of R iff

$t_1, \ldots, t_n \in R$ such that $(t_1[i], \ldots, t_n[i]) \in E$ for every i implies

$$\begin{pmatrix} f(t_1[1], \ldots, t_n[1]) \\ \vdots \\ f(t_1[k], \ldots, t_n[k]) \end{pmatrix} \in R$$

$\mathrm{pPol}(R)$ is set of all partial polymorphisms.

$\mathrm{pPol}(R)$ is strongly partial clone.

The Galois Correspondence

pPol and Inv form a Galois Correspondence [Romov 1978]

- $\mathrm{Inv}(\mathrm{pPol}(\Gamma)) = \langle \Gamma \rangle_{\nexists}$
- $\mathrm{pPol}(\mathrm{Inv}(B)) = [B]_{\mathrm{par}}$

No "Post's Lattice" for partial clones.

"Easy" Relations

Obtain simple proof for complexity classification of ENUM(Γ).

Difficult part: hardness results for non Schaefer languages.

Find "easiest" $R \in \langle \Gamma \rangle$ such that

- $\langle R \rangle = \langle \Gamma \rangle$
- ENUM(R) reduces easily to ENUM(Γ') for all $\langle \Gamma' \rangle = \langle \Gamma \rangle$,

  i.e. $R \in \langle \Gamma' \rangle_{\nexists}$

Extensions

$$R = \begin{pmatrix} (t_1^1 & \dots & t_r^1) \\ (t_1^2 & \dots & t_r^2) \\ & \vdots & \\ (t_1^n & \dots & t_r^n) \end{pmatrix}$$

Extensions

$$R^{\text{ext}} \approx \begin{pmatrix} (\overbrace{t_1^1 & \dots & t_r^1}^{R} & 0 & 1 & 0 & \dots & 1) \\ (t_1^2 & \dots & t_r^2 & 0 & 0 & 1 & \dots & 1) \\ & \vdots & & & & \vdots & & \\ (t_1^n & \dots & t_r^n & 0 & 0 & 0 & \dots & 1) \end{pmatrix}$$

$\mathrm{Pol}(R^{\text{ext}}) = \mathrm{Pol}(R)$

- $\mathrm{pPol}(\Gamma) \subseteq \mathrm{pPol}(R^{\text{ext}})$ for all $\mathrm{Pol}(\Gamma) = \mathrm{Pol}(R)$.
- $R^{\text{ext}} \in \langle \Gamma \rangle_{\nexists}$ for all $\langle \Gamma \rangle = \langle R \rangle$.

CSP*

CSP*(Γ):

Input: Γ-formula ϕ

Question: Does ϕ have a non-constant solution?

If CSP*(Γ) is NP-hard then ENUM(Γ) is not efficient.

Enumeration and CSP^*

If Γ is not Schaefer, then $\mathrm{CSP}^*(\Gamma)$ is NP-hard.

- Let R such that $\langle R \rangle = \langle \Gamma \rangle$
- $\mathrm{CSP}^*(R^{\mathrm{ext}}) \leq_m^P \mathrm{CSP}^*(\Gamma)$
- $\mathrm{CSP}_{\mathrm{c}}(R)$ is NP-hard [Schaefer 1978]
- Sufficient to show: $\mathrm{CSP}_{\mathrm{c}}(R) \leq_m^P \mathrm{CSP}^*(R^{\mathrm{ext}})$

$\mathrm{CSP_c}(R) \leq_m^P \mathrm{CSP}^*(R^{\mathrm{ext}})$

$$\phi = \bigwedge_i R(x_1^i, \ldots, x_n^i) \qquad x_j^i \in \mathrm{Var} \cup \{0, 1\}$$

$$\Big\downarrow \begin{array}{l} x_j^i = 0 \text{ set } x_j^i = f \\ x_j^i = 1 \text{ set } x_j^i = t \end{array}$$

$$\psi = \bigwedge_i R^{\mathrm{ext}}(x_1^i, \ldots, x_n^i, f, y_1^i, \ldots, y_\ell^i, t)$$

$$\phi \in \mathrm{CSP_c}(R) \Rightarrow \psi \in \mathrm{CSP}^*(R^{\mathrm{ext}})$$

$$R^{\mathrm{ext}} \approx \begin{pmatrix} (t_1^1 & \ldots & t_r^1 & 0 & 1 & \ldots & 1) \\ (t_1^2 & \ldots & t_r^2 & 0 & 0 & \ldots & 1) \\ & \vdots & & & & \vdots & \\ (t_1^n & \ldots & t_r^n & 0 & 0 & \ldots & 1) \end{pmatrix}$$

$\mathrm{CSP_c}(R) \leq_m^P \mathrm{CSP}^*(R^{\mathrm{ext}})$

$$\phi = \bigwedge_i R(x_1^i, \ldots, x_n^i) \qquad x_j^i \in \mathrm{Var} \cup \{0,1\}$$

$$\Big\downarrow \begin{array}{l} x_j^i = 0 \text{ set } x_j^i = f \\ x_j^i = 1 \text{ set } x_j^i = t \end{array}$$

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$$\phi \in \mathrm{CSP_c}(R) \Leftarrow \psi \in \mathrm{CSP}^*(R^{\mathrm{ext}})$$

$$R^{\mathrm{ext}} \approx \begin{pmatrix} (t_1^1 & \ldots & t_r^1 & 0 & 1 & \ldots & 1) \\ (t_1^2 & \ldots & t_r^2 & 0 & 0 & \ldots & 1) \\ & \vdots & & & & \vdots & \\ (t_1^n & \ldots & t_r^n & 0 & 0 & \ldots & 1) \end{pmatrix}$$

$$\mathrm{Pol}(R^{\mathrm{ext}}) = \mathrm{Pol}(R) \subseteq \{\mathrm{id}, \neg, 0, 1\}$$

If Γ not Schaefer, then ENUM(Γ) is not efficient.

Similar re-proof for EQ(Γ).

"Hardest" bases identified as well. [Creignou, Kolaitis, Zanuttini 2005]

Idea: Use method to see when Pol-Inv-correspondence works:

- Does "hardest" constraint language from $\langle \Gamma \rangle$ reduce to "easiest"?