Complexity of Maximum Solution:
Max-Ones(Γ) Generalised to Larger Domains

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The Maximum Solution Problem: Max-Sol(Γ)

**Input:** CSP(Γ) (domain $D \subseteq \mathbb{N}$) instance $I$ on variables $V$

**Objective:** Find a solution $s$ to $I$ such that

$$\sum_{v \in V} s(v)$$

is maximised.
The Maximum Solution Problem: Max-Sol(Γ)

**Input:** CSP(Γ) (domain \( D \subseteq \mathbb{N} \)) instance \( I \) on variables \( V \) and a weight function \( w : V \rightarrow \mathbb{N} \)

**Objective:** Find a solution \( s \) to \( I \) such that

\[
\sum_{v \in V} w(v) \cdot s(v)
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**Example**

$(x + y \leq 1)$ over $D = \{0, 1\}$ with $w(x) = 2$, $w(y) = 1$ have the optimal solution $s(x) = 1$, $s(y) = 0$
The Maximum Solution Problem: Max-Sol($\Gamma$)

**Input:** CSP($\Gamma$) (domain $D \subseteq \mathbb{N}$) instance $I$ on variables $V$ and a weight function $w : V \rightarrow \mathbb{N}$

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**Example**

$R_{\neg x \lor \neg y} = \{(0, 0), (0, 1), (1, 0)\}$

Max-Sol($R_{\neg x \lor \neg y}$) is Maximum Independent Set.
**Input:** CSP(\(\Gamma\)) (domain \(D \subseteq \mathbb{N}\)) instance \(I\) on variables \(V\) and a weight function \(w : V \to \mathbb{N}\)

**Objective:** Find a solution \(s\) to \(I\) such that

\[
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is maximised.

- Max-Sol(\(\Gamma\)) over \(D = \{0, 1\}\) is called Max-Ones(\(\Gamma\))
  - The complexity of Max-Ones(\(\Gamma\)) is known for all \(\Gamma\) [Khanna, Sudan, Trevisan, Williamson 00]

- Our goal is to classify the complexity of Max-Sol(\(\Gamma\)) for all \(\Gamma\)
**Input:** matrix $A$, a vector of variables $x = (x_1, \ldots, x_n)^t$, a column vector $b$ and a sequence of numbers $c_1, \ldots, c_n$.

**Objective:** Maximise

$$\sum_{i=1}^{n} c_i x_i$$

subject to $Ax \leq b$, $x_i$’s non-negative integers $\leq B$. 

$\Gamma_{\text{LIN}}$: Linear inequalities over $D = \{0, \ldots, B\}$

Max-Sol($\Gamma_{\text{LIN}}$) is very close to B-ILP.
Motivation: (Bounded) Integer Linear Programming (B-ILP)

Input: matrix $A$, a vector of variables $\mathbf{x} = (x_1, \ldots, x_n)^t$, a column vector $\mathbf{b}$ and a sequence of numbers $c_1, \ldots, c_n$.

Objective: Maximise

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subject to $A\mathbf{x} \leq \mathbf{b}$, $x_i$'s non-negative integers $\leq B$.

- $\Gamma_{LIN}$: Linear inequalities over $D = \{0, \ldots, B\}$
- $\text{Max-Sol}(\Gamma_{LIN})$ is very close to B-ILP.
The complexity of $\text{Max-Sol}(\Gamma)$ is determined by $\text{Pol}(\Gamma)$

The complexity of $\text{Max-Ones}(\Gamma)$ revisited

Tractable class for $\text{Max-Sol}(\Gamma)$

$\Gamma$ expressed by Regular Signed Logic

Maximal constraint languages
A is an $r$-approximate algorithm for a (maximisation) problem if

$$\frac{\text{OPT}(I)}{\text{A}(I)} \leq r$$

- **PO**: Optimal solutions in polynomial time
- **APX**: Polynomial time $r$-approximate algorithm for some constant $r \geq 1$
- **poly-APX**: Polynomial time $r$-approximate algorithm for some $r \leq p(|I|)$

Hardness/completeness for (poly-)APX is defined in terms of approximation preserving reductions.
Theorem

If $\Gamma_1 \subseteq \langle \Gamma_2 \rangle$, then there is an approximation preserving reduction from Max-Sol($\Gamma_1$) to Max-Sol($\Gamma_2$).
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Given an instance $I$ of Max-Sol($\Gamma_1$):

$R_1(x, \ldots, y) \equiv \exists z R_2(x, \ldots, z) \land \cdots \land R_2(x, \ldots, y) \land (x = y)$

- Eliminate existential quantifiers by introducing fresh variables of weight 0
- Eliminate $x = y$ by replacing $y$ by $x$ and adding $w(y)$ to $w(x)$
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- Eliminate existential quantifiers by introducing fresh variables of weight 0
- Eliminate $x = y$ by replacing $y$ by $x$ and adding $w(y)$ to $w(x)$
- For the resulting instance $I'$ of Max-Sol($\Gamma_2$) we have
  - $\text{OPT}(I) = \text{OPT}(I')$
  - For any solution $s'$ to $I'$ we can find in polynomial time a solution $s$ to $I$ such that $\text{Value}(s) = \text{Value}(s')$. 
Theorem

If \( \Gamma_1 \subseteq \langle \Gamma_2 \rangle \), then there is an \textit{S-reduction} from Max-Sol(\(\Gamma_1\)) to Max-Sol(\(\Gamma_2\)).
Algebraic approach to Max-Sol($\Gamma$)

Theorem

If $\Gamma_1 \subseteq \langle \Gamma_2 \rangle$, then there is an $S$-reduction from Max-Sol($\Gamma_1$) to Max-Sol($\Gamma_2$).

Corollary

If $\text{Pol}(\Gamma_1) = \text{Pol}(\Gamma_2)$ then Max-Sol($\Gamma_1$) is in PO (APX-complete, poly-APX-complete) iff Max-Sol($\Gamma_2$) is in PO (APX-complete, poly-APX-complete)
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Observation

If $\text{Pol}(\Gamma_1) = \text{Pol}(\Gamma_2)$ and Max-Sol($\Gamma_1$) is in APX and approximable within $a$ but NP-hard to approximate within $b$ then Max-Sol($\Gamma_2$) is approximable within $a$ but NP-hard to approximate within $b$. 
Max-Ones(\(\Gamma\)) revisited

Theorem ([Khanna, Sudan, Trevisan, Williamson 00])

Max-Ones(\(\Gamma\)) is either in \(\text{PO}\), or \(\text{APX}\)-complete, or \(\text{poly-APX}\)-complete, or finding a positive solution is \(\text{NP}\)-hard, or finding a feasible solution is \(\text{NP}\)-hard.
Max-Ones(Γ) revisited
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Conjecture

Max-Ones(Γ) is either in \text{PO}, or approximable within 2 and \text{NP}-hard to approximate within $2 - \varepsilon$ for any $\varepsilon > 0$, or approximable within $O(n)$ and \text{NP}-hard to approximate within $O(n^{1-\varepsilon})$ for any $\varepsilon > 0$, or finding a positive solution is \text{NP}-hard, or finding a feasible solution is \text{NP}-hard.
Definition

Γ is **generalised max-closed** if there exists a $f \in Pol(\Gamma)$ such that for all $a < b \ (a, b \in D)$ we have $f(a, b) > a$, $f(b, a) > a$, and for all $a \in D$ we have $f(a, a) \geq a$. 

Theorem

If Γ is generalised max-closed, then Max-Sol(Γ) is in PO.
Definition

Γ is generalised max-closed if there exists a \( f \in Pol(\Gamma) \) such that for all \( a < b \) (\( a, b \in D \)) we have \( f(a, b) > a \), \( f(b, a) > a \), and for all \( a \in D \) we have \( f(a, a) \geq a \).

Theorem

If Γ is generalised max-closed, then Max-Sol(Γ) is in PO.
Lemma

If $\Gamma$ is generalised max-closed, then all relations

$$R = \{(d_{11}, d_{12}, \ldots, d_{1m}), \ldots, (d_{t1}, d_{t2}, \ldots, d_{tm})\}$$

in $\Gamma$ have the property that the tuple

$$\mathbf{t}_{\text{max}} = (\max\{d_{11}, \ldots, d_{t1}\}, \ldots, \max\{d_{1m}, \ldots, d_{tm}\})$$

is in $R$, too.
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Given an instance $I$ of Max-Sol($\Gamma$) where $\Gamma$ is generalised max-closed:

- Establish pairwise consistency
- Assigning to each variable $x$ the maximum value it is allowed to take by any constraint $C(\ldots, x, \ldots)$ is a solution to $I$. 
Constraints are disjunctions of inequalities over a (totally ordered) domain $D = \{0, \ldots, d\}$.

Example

$D = \{0, 1, 2\}$,

$(x \leq 0 \lor y \leq 1) \equiv \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (2, 0), (2, 1)\}$
Constraints are disjunctions of inequalities over a (totally ordered) domain $D = \{0, \ldots, d\}$.

**Example**

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**Example**

$D = \{0, 1\}$, $\text{Max-Sol}(x \leq 0 \lor y \leq 0)$ is Maximum Independent Set.
Constraints are disjunctions of inequalities over a (totally ordered) domain \( D = \{0, \ldots, d\} \).

**Example**

\[ D = \{0, 1, 2\}, \]
\[ (x \leq 0 \lor y \leq 1) \equiv \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (2, 0), (2, 1)\} \]

**Example**

\[ D = \{0, 1\}, \text{ Max-Sol}(x \leq 0 \lor y \leq 0) \text{ is Maximum Independent Set.} \]

**Example**

\[ D = \{0, 1\}, \text{ Min-Sol}(x \geq 1 \lor y \geq 1) \text{ is Minimum Vertex Cover,} \]
$\Gamma_L$ denote constraint languages expressed by regular signed logic clauses

**Theorem ([Creignou, Hermann, Krokhin, Salzer 06])**

*For any $\Gamma_L$ CSP($\Gamma_L$) is either in P or NP-complete.*
Theorem

Max-Sol(Γ_L) is in PO if Γ_L is generalised max-closed and APX-hard otherwise.
Max-Sol($\Gamma_L$): Classification

Theorem

$\text{Max-Sol}(\Gamma_L)$ is in $\text{PO}$ if $\Gamma_L$ is generalised max-closed and $\text{APX}$-hard otherwise.

- $\Gamma_L$ is generalised max-closed: DONE!
- $\Gamma_L$ not generalised max-closed implies that $(x \leq a \lor y \leq b) \in \langle \Gamma_L \rangle$ where $a, b < \max\{D\}$
- $\text{APX}$-hardness of $\text{Max-Sol}(x \leq a \lor y \leq b)$ by reduction from $\text{MAX}-3\text{SAT}$
Min-Sol($\Gamma_L$): Classification

**Theorem**

*Min-Sol*($\Gamma_L$) *is in* PO *if* $\Gamma_L$ *is generalised min-closed and APX-hard otherwise.*
Min-Sol(\(\Gamma_L\)): Classification

**Theorem**

\[\text{Min-Sol}(\Gamma_L) \text{ is in PO if } \Gamma_L \text{ is generalised min-closed and APX-hard otherwise.}\]

- \(\Gamma_L\) generalised min-closed implies that Min-Sol(\(\Gamma_L\)) in **PO** by the generalised max-closed algorithm
- APX-hardness by reduction from Minimum Vertex Cover
AW-Max-Sol(Γ_L) (Arbitrary Weights are allowed): Classification

Theorem

AW-Max-Sol(Γ_L) is in PO if \{\text{min}, \text{max}\} \subseteq Pol(Γ_L) and APX-hard otherwise.
AW-Max-Sol($\Gamma_L$) (Arbitrary Weights are allowed): Classification

**Theorem**

$AW\text{-}Max\text{-}Sol(\Gamma_L)$ is in PO if $\{\min, \max\} \subseteq Pol(\Gamma_L)$ and APX-hard otherwise.

- $\{\min, \max\} \subseteq Pol(\Gamma_L)$ implies that $AW\text{-}Max\text{-}Sol(\Gamma_L)$ can be solved by maximising a supermodular function
- Hardness by reductions from $Max\text{-}Sol(\Gamma_L)$ and $Min\text{-}Sol(\Gamma_L)$
Definition

Γ is a maximal constraint language over $D$ if $\langle \Gamma \rangle \subset ALL_D$ and for any $R \notin \langle \Gamma \rangle$ we have $\langle \Gamma \cup \{R\} \rangle = ALL_D$. 

Theorem ([Bulatov, Krokhin, Jeavons 01], [Bulatov 04])

If $\Gamma$ is a maximal constraint language, then CSP($\Gamma$) is either in $P$ or $NP$-complete. 

Theorem

If $\Gamma$ is a maximal constraint language, then Max-Sol($\Gamma$) is either in $PO$, or $APX$-complete, or poly-$APX$-complete, or finding a positive solution is $NP$-hard, or finding a feasible solution is $NP$-hard.
**Definition**

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**Theorem ([Bulatov, Krokhin, Jeavons 01], [Bulatov 04])**

*If \( \Gamma \) is a maximal constraint language, then CSP(\( \Gamma \)) is either in \( \mathbf{P} \) or \( \mathbf{NP} \)-complete.*
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### Theorem

*If Γ is a maximal constraint language, then Max-Sol(Γ) is either in PO, or APX-complete, or poly-APX-complete, or finding a positive solution is NP-hard, or finding a feasible solution is NP-hard.*
Let $\Gamma = \text{Inv}(\{f\})$ be a maximal constraint language on $D \subseteq \mathbb{N}$.

1. If $\Gamma$ is generalised-max-closed, then $\text{MAX SOL}(\Gamma)$ is in $\text{PO}$;
2. else if $f$ is an affine operation, a constant operation different from the constant 0 operation, or a binary commutative idempotent operation satisfying $f(0, b) > 0$ for all $b \in D \setminus \{0\}$; or if $0 \notin D$ and $f$ is a binary commutative idempotent operation or a majority operation, then $\text{MAX SOL}(\Gamma)$ is $\text{APX}$-complete;
3. else if $f$ is a binary commutative idempotent operation or a majority operation, then $\text{MAX SOL}(\Gamma)$ is $\text{poly-APX}$-complete;
4. else if $f$ is the constant 0 operation, then finding a positive solution is $\text{NP}$-hard;
5. otherwise, finding a feasible solution is $\text{NP}$-hard.
CONJECTURE: Max-Sol(Γ) is either in \textbf{PO}, or \textbf{APX}-complete, or \textbf{poly-APX}-complete, or finding a positive solution is \textbf{NP}-hard, or finding a feasible solution is \textbf{NP}-hard.
Ladner’s theorem and CSP(Γ)
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Max-Sol(Γ) and PTAS
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Open

- Max-Sol(Γ) where Γ is a graph
- Max-Sol(Γ) over 3-element domains