Non-FO CSP's are Logspace-hard

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Outline

Dichotomies

- Boolean Case: a complete description of the complexity of Boolean CSP's by Allender, Bauland, Immerman, Schnoor, Vollmer 2005
- the quest for the **P** vs **NP**-complete dichotomy for CSP's: let's try a simpler dichotomy first !

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Work in progress

- we prove that any CSP which is not first-order (FO) definable is LOGSPACE hard;
- our program: we are presently pursuing an attempt to describe certain CSP's in LOGSPACE via special Datalog programs, special operations, etc. ...

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Basic Definitions

- Fix a relational language σ;
  STRUCT(σ) is the class of all finite structures of type σ;
- fix a relational structure $\mathbb{H} \in$ *STRUCT*(σ):
  CSP$(\mathbb{H})$ is the class of all structures of type σ that admit a homomorphism to $\mathbb{H}$.
- wlog we assume that $\mathbb{H}$ is a *core*, i.e. admits no proper endomorphisms.

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FO CSP's

- *CSP*$(\mathbb{H})$ is *first-order (FO) definable* if there exists a FO sentence Φ in the language σ such that

$$\Phi \text{ valid in } \mathbb{K} \Leftrightarrow \mathbb{K} \rightarrow \mathbb{H}$$

- a.k.a. has finite duality (Atserias 2005),
- a.k.a complement is definable in non-recursive DATALOG,
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FO reductions

- let τ denote the type $\{s, t, E\}$ where s and t are constant symbols and E is a binary relation symbol;
- τ is the language of graphs with two distinguished vertices;
- we shall be interested in those structures in $STRUCT(\tau)$ which are symmetric graphs (with two distinguished vertices)

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FO reductions, cont'd

- fix a core $\mathbb{H}$ of type σ with $CSP(\mathbb{H})$ not FO;
- the reduction is in two distinct steps, one of which allows us to assume the following technicality:
- the type σ has a unary relational symbol S_h for each $h \in H$ such that $S_h(\mathbb{H}) = \{h\}$ for all $h \in H$.

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FO reductions, cont'd

- the main step of the reduction is a map

$$\mathcal{I} : \mathit{STRUCT}(\tau) \longrightarrow \mathit{STRUCT}(\sigma)$$

  such that
  - for every graph $\mathbb{G}$, s and t are not connected $\Leftrightarrow \mathcal{I}(\mathbb{G}) \rightarrow \mathbb{H}$;
  - the query is first order, i.e. the structure $\mathcal{I}(\mathbb{G})$ is first-order defined in terms of $\mathbb{G}$.

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Some intermediate complexity classes

LOGSPACE

NC^1 (logarithmic depth circuits)

TC^0 (majority gates)

ACC^0 (+ MOD gates)

(strict inclusion: parity)

$FO = AC^0$ (First order definable) (constant depth circuits)

Links

Definition

The *n-link* of type σ is the σ-structure

$$\mathbb{L}_n = \langle \{0, 1, \ldots, n\}; R_1(\mathbb{L}_n), \ldots, R_r(\mathbb{L}_n) \rangle,$$

such that $R_i(\mathbb{L}_n) = \cup_{j=1}^{n} \{j-1, j\}^{a_i}$ for $i = 1, \ldots, r$.

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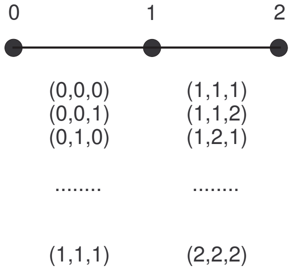
Links, cont'd

- Restricted to each pair of consecutive integers, each relation $R_i(\mathbb{L}_n)$ is "full".
- Below, the link $\mathbb{L}_2$ with one ternary relation shown:



Powers

Definition (Lovász, 1967)

Let $\mathbb{A}$ and $\mathbb{B}$ be two σ-structures. The σ-structure $\mathbb{B}^{\mathbb{A}}$ has universe $\{f : A \to B\}$ and is designed to satisfy the following property: for every $\mathbb{C}$ we have

$$\mathbb{C} \to \mathbb{B}^{\mathbb{A}} \Leftrightarrow \mathbb{C} \times \mathbb{A} \to \mathbb{B}.$$

A Criterion for FO definability

Theorem (BL, Loten, Tardif 2006)

Let $\mathbb{H}$ be a core structure. Then CSP$(\mathbb{H})$ is FO-definable if and only if there exists a homomorphism from some link $\mathbb{L}_n$ to $\mathbb{H}^{\mathbb{H}^2}$ mapping the endpoints to the two projections.

A Criterion for FO definability, cont'd

I.e. viewing the structure $\mathbb{H}^{\mathbb{H}^2}$ as a graph where $\{f, g\}$ forms an edge if the link $\mathbb{L}_1$ maps to $\{f, g\}$,

$CSP(\mathbb{H})$ is NOT FO-definable

$\Leftrightarrow$

the projections are not connected in this graph.

The reduction

- Let G be a graph with specified vertices s and t;
- construct the σ-structure Γ from G by replacing each edge by the link $\mathbb{L}_1$;
- construct the σ-structure $\mathbb{P} = \Gamma \times \mathbb{H}^2$, imposing further the conditions

$$(s, c, d) \in S_c \text{ and } (t, c, d) \in S_d$$

for all $(c, d) \in H^2$.

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The reduction, cont'd

- it follows that $\mathbb{P} \longrightarrow \mathbb{H}$ precisely if $\Gamma \longrightarrow \mathbb{H}^{\mathbb{H}^2}$ in such a way that s and t map to the two projections;
- this is possible if and only if s and t are NOT connected in G.
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