Constraint satisfaction problems and dualities

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The CSP

The Constraint Satisfaction (or Homomorphism) Problem:

Given two finite relational structures,
$\mathcal{A} = (A;\ R_1^{\mathcal{A}}, \ldots, R_m^{\mathcal{A}})$ and $\mathcal{B} = (B;\ R_1^{\mathcal{B}}, \ldots, R_m^{\mathcal{B}})$,
is there a homomorphism $h : \mathcal{A} \to \mathcal{B}$?

$\forall i\ [(a_1, \ldots, a_n) \in R_i^{\mathcal{A}} \Longrightarrow (h(a_1), \ldots, h(a_n)) \in R_i^{\mathcal{B}}]$

If such an h exists, we write $\mathcal{A} \to \mathcal{B}$.

The CSP with a fixed $\mathcal{B}$ is denoted CSP($\mathcal{B}$).

One can view CSP($\mathcal{B}$) as $\{\mathcal{A} \mid \mathcal{A} \to \mathcal{B}\}$.

Classification problems

Two main classification problems about problems CSP($\mathcal{B}$):

1. Classify CSP($\mathcal{B}$) w.r.t. computational complexity
2. Classify CSP($\mathcal{B}$) w.r.t. descriptive complexity, (i.e., definability in a given logic)

In addition, there is a meta-problem:

- Determine the complexity of deciding whether CSP($\mathcal{B}$) has given (computational or descriptive) complexity.

In this talk - a simple combinatorial idea that has a bearing on all the above problems.

Obstruction sets

Obvious fact: if $\mathcal{A}' \to \mathcal{A}$ and $\mathcal{A}' \not\to \mathcal{B}$ then $\mathcal{A} \not\to \mathcal{B}$.

Definition 1 *An obstruction set for a structure $\mathcal{B}$ is a class $\mathcal{O}_\mathcal{B}$ of structures such that, for all structures $\mathcal{A}$,*

$$\mathcal{A} \to \mathcal{B} \text{ if and only if } \mathcal{A}' \not\to \mathcal{A} \text{ for all } \mathcal{A}' \in \mathcal{O}_\mathcal{B}.$$

Example 1

- *If $\mathcal{B}$ is a transitive tournament $\vec{T}_k$ on k vertices then one can choose $\mathcal{O}_\mathcal{B} = \{\vec{P}_k\}$ where $\vec{P}_k$ is a directed path on $k+1$ vertices.*
- *If $\mathcal{B}$ is a bipartite graph then $\mathcal{O}_\mathcal{B}$ can be chosen to consist of all odd cycles.*

Dualities

1. Finite duality
   $\mathcal{O}_{\mathcal{B}}$ can be chosen finite
   Example: $\mathcal{B}$ is a transitive tournament $\vec{T}_k$

2. Bounded pathwidth duality
   $\mathcal{O}_{\mathcal{B}}$ can be chosen of bounded pathwidth
   Example: $\mathcal{B}$ is a bipartite (undirected) graph

3. Bounded treewidth duality
   $\mathcal{O}_{\mathcal{B}}$ can be chosen of bounded treewidth
   Example: HORN k-SAT, $k \geq 3$.

Trees and paths

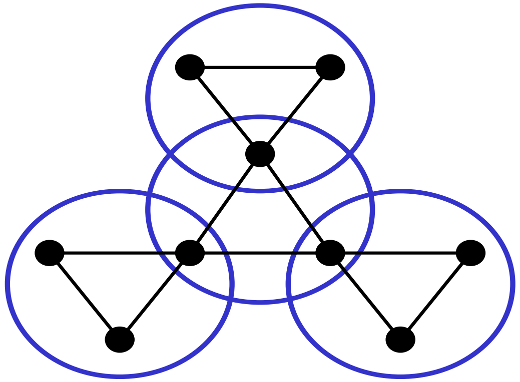
A structure $\mathcal{D}$ is an (l,k)-tree if there is a tree T such that

- the nodes of T are subsets of D of size $\leq k$
- adjacent nodes can share $\leq l$ elements
- nodes containing any given element form a subtree
- for any tuple in any relation in $\mathcal{D}$, there is a node in T containing all elements from that tuple.

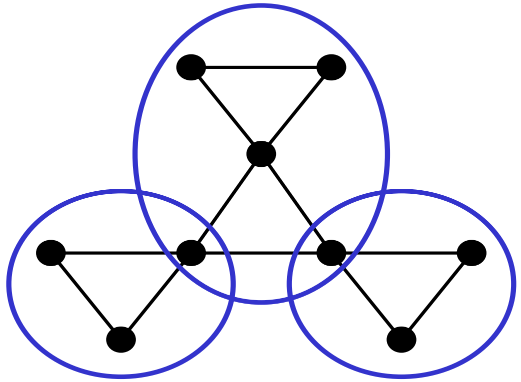
If T is a path then $\mathcal{D}$ is called an (l,k)-path.

NB. Often (but not in this talk), (l,k)-trees are called structures of treewidth at most $k-1$.

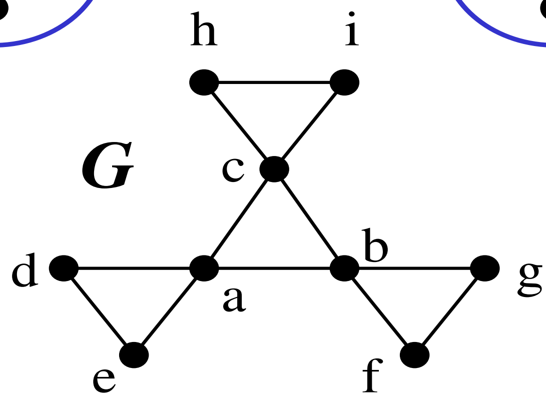
Example



G as (1,3)-tree

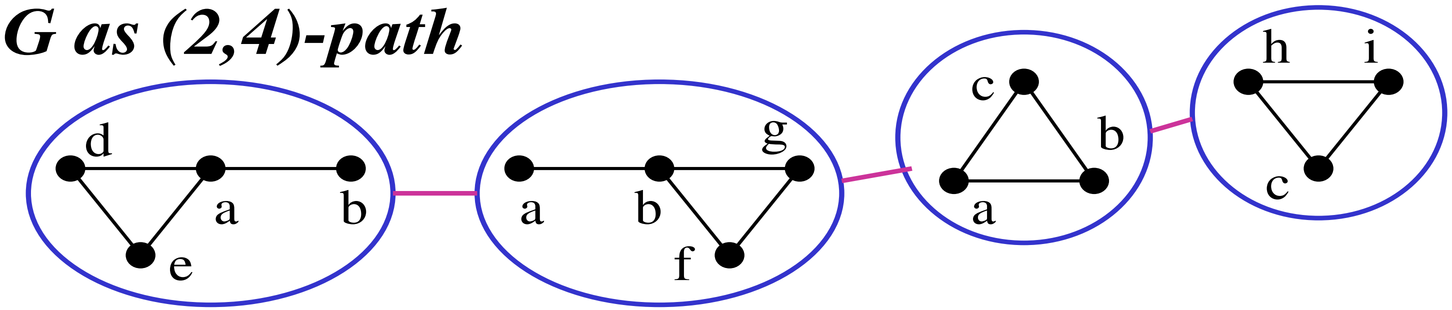


G as (1,5)-path



G

G as (2,4)-path



Treewidth and pathwidth dualities

Definition 2 *A structure $\mathcal{B}$ is said to have*

- *(l,k)-treewidth duality if $\mathcal{B}$ has an obstruction set consisting of (l,k)-trees,*
- *l-treewidth duality if $\mathcal{B}$ has (l,k)-treewidth duality for some $k > l$,*
- *bounded treewidth duailty if $\mathcal{B}$ has l-treewidth duality for some l.*

Pathwidth dualities are defined similarly.

Retract and core

Definition 3 *A retract of a structure $\mathcal{B}$ is an induced substructure $\mathcal{B}'$ of $\mathcal{B}$ such that there is an endomorphism g of $\mathcal{B}$ with $Im(g) = B'$ and $g(b) = b$ for every $b \in B'$.*

A retract of $\mathcal{B}$ is called a core of $\mathcal{B}$ if it has minimal size among all retracts.

Basic facts:

- All cores of $\mathcal{B}$ are isomorphic.
- If $\mathcal{B}'$ is a retract of $\mathcal{B}$ then $\mathrm{CSP}(\mathcal{B}) = \mathrm{CSP}(\mathcal{B}')$.

Clearly, a structure has a given duality iff its core does.

What's to come

In this talk:

- logical descriptions of the three dualities
- algebraic conditions for structures with a given duality
- computational complexity of the meta-problem

Datalog

For logical descriptions, we use FO and (linear) Datalog.

$$
\begin{aligned}
oddpath(X,Y) \quad &:- \quad edge(X,Y) \\
oddpath(X,Y) \quad &:- \quad oddpath(X,Z), edge(Z,T), edge(T,Y) \\
non2colorable \quad &:- \quad oddpath(X,X)
\end{aligned}
$$

An (l,k)-Datalog program is a Datalog program with $\leq l$ variables per rule head and $\leq k$ variables per rule.

EDBs = relations from structure, IDBs = new relations.
A Datalog program is called linear if every rule has at most one IDB in its body.

Bounded treewidth duality

Theorem 1 (Feder,Vardi'98) *T.f.a.e.:*

1. $\mathcal{B}$ *has* (l, k)*-treewidth duality.*
2. $\neg\mathrm{CSP}(\mathcal{B})$ *is definable in* (l, k)*-Datalog.*

If these conditions hold then $\mathrm{CSP}(\mathcal{B})$ *is in* **PTIME**.

Kolaitis and Vardi provided various equivalent conditions, including definability in certain infinitary finite-variable logics and a pebble-game characterisation.

Tree duality

Tree duality is just a shorter name for 1-treewidth duality.

Theorem 2 (Feder,Vardi'98)
1) For each $\mathcal{B}$, there is an effectively constructible finite structure $\mathcal{P}_1(\mathcal{B})$ such that $\mathcal{B}$ has tree duality iff $\mathcal{P}_1(\mathcal{B}) \to \mathcal{B}$.

2) A structure $\mathcal{B}$ has tree duality iff, for every n, $\mathcal{B}$ has an n-ary polymorphism f_n such that $f(x_1, , \ldots, x_n)$ depends only on $\{x_1, \ldots, x_n\}$.

Example: Horn k-clauses have polym. $f_n = \bigwedge_{i=1}^{n} x_i$.

Theorem 3 (Larose,Loten,Tardif'06) *It is* **NP***-hard to decide whether a given structure $\mathcal{B}$ has tree duality.*

What else has BTW duality

A near-unanimity (NU) operation is an n-ary ($n \geq 3$) operation such that, for all x, y,

$$f(y, x, \ldots, x) = \ldots = f(x, \ldots, x, y) = x.$$

Theorem 4 (Feder,Vardi'98) *If $\mathcal{B}$ has a (l+1)-ary NU polymorphism then $\mathcal{B}$ has l-treewidth duality.*

A binary idempotent commutative operation f is called a 2-semilattice operation if $f(x, f(x, y)) = f(x, y)$ for all x, y.

Theorem 5 (Bulatov'05) *If $\mathcal{B}$ has a 2-semilattice polymorphism then $\mathcal{B}$ has bounded treewidth duality.*

What has no BTW duality

Fact. LIN EQ has no BTW duality.

Intuition:

1. If a CSP is solvable by an (l, k)-Datalog program then it can be solved by looking at small parts of $\mathcal{A}$, one at a time.
2. If LIN EQ were solvable by such a program, this would mean that linear systems can be solved by inspecting parts of such systems involving at most k variables at a time.
3. It can be easily checked that linear systems cannot be solved like that.

Conjecture

A weak NU operation is an n-ary ($n \geq 3$) operation such that, for all x, y, we have $f(x, \ldots, x) = x$ and

$$f(y, x, \ldots, x) = \ldots = f(x, \ldots, x, y).$$

Conjecture 1 (Bulatov; Larose,Zádori)
A core structure $\mathcal{B}$ has BTW duality iff one of the following equivalent conditions holds:

- *The variety* $\mathbf{var}(\mathbf{A}_{\mathcal{B}})$ *omits types* **1** *and* **2**;
- $\mathcal{B}$ *has* n*-ary weak NU polymorphisms for all* $n \geq 2|B|!$. *(recently added by Mároti and McKenzie).*
- $\mathcal{B}$ *has a Taylor term, and* $Gr(\mathcal{B})$ *has no blue edges.*

Facts about the conjecture

Theorem 6 (Bulatov; Larose,Zádori) *The "only if" part of the conjecture holds.*

Theorem 7 (Bulatov) *The conjecture holds if $\mathcal{B}$ has all unary relations or if $|B| \leq 3$.*

Theorem 8 (Bulatov) *Assuming the conjecture holds, the meta-problem for BTW is* **NP**-*complete for arbitrary structures, but tractable for structures of bounded size.*

Theorem 9 (Bulatov; Larose,Valeriote) *There is a polynomial time algorithm which, given a finite algebra* $\mathbf{A}$, *checks whether the variety* $\mathbf{var}(\mathbf{A})$ *omits types* $\mathbf{1}$ *and* $\mathbf{2}$.

Hierarchy of treewidth dualities

Let TW_l be the class of structures with l-treewidth duality.

Fact. $TW_1 \subsetneq TW_2 \subseteq TW_3 \subseteq TW_4 \subseteq TW_5 \subseteq \ldots$.

Open question. Does this hierarchy collapse or not?

Bounded pathwidth duality

Theorem 10 (Dalmau'05) *T.f.a.e.:*

1. $\mathcal{B}$ *has* (l, k)*-pathwidth duality.*
2. $\neg\mathrm{CSP}(\mathcal{B})$ *is definable in linear* (l, k)*-Datalog.*

If these conditions hold then $\mathrm{CSP}(\mathcal{B})$ *is in* **NL**.

Dalmau also provides various equivalent conditions, including definability in certain infinitary finite-variable logics and a pebble-game characterisation.

What has BPW duality

Theorem 11 (Dalmau'05, Dalmau,K.'06)

- *Let* $|B| = 2$. *Then* $\mathcal{B}$ *has BPW duality iff* $\mathcal{B}$ *has a NU polymorphism.*
- *If* $|B| = k$ *and* $\mathcal{B}$ *has a majority (i.e., 3-ary NU) polymorphism then* $\mathcal{B}$ *has* $(3k+2)$*-pathwidth duality.*

In general, all problems CSP($\mathcal{B}$) known to be in **NL** have BPW duality.

K. and Larose gave examples of CSP($\mathcal{B}$) with BPW duality such that $\mathcal{B}$ has no NU polymorphism of any arity.

What has no BPW duality

- Clearly, any structure without BTW duality cannot have BPW duality.
- For $k \geq 3$, Horn k-Sat has no BPW duality.
- Conjecture? Too early.

Hierarchy of pathwidth dualities

Theorem 12 (Dalmau,K.'06) *For every $n \geq 2$, there is a structure $\mathcal{B}_n$ such that*

- $|B_n| = 2n$ *and* $\mathcal{B}_n$ *has* n *binary relations,*
- $\mathcal{B}_n$ *has no* n*-pathwidth duality,*
- $\mathcal{B}_n$ *has a majority polymorphism (and hence* $(6n+2)$*-pathwidth duality).*

Hence the hierarchy of pathwidth dualities does not collapse.

Finite duality

Clearly, finite duality for $\mathcal{B}$ implies that CSP($\mathcal{B}$) is FO-definable.

Theorem 13 (Atserias'05,Rossman'05) *A structure $\mathcal{B}$ has finite duality iff* CSP($\mathcal{B}$) *is* FO-*definable.*

Theorem 14 (Nešetřil,Tardif'00) *A finite set $\mathcal{O}$ of structures is an obstruction set for some structure $\mathcal{B}$ iff the incidence multigraph of every structure in $\mathcal{O}$ is a tree. If this condition holds then some structure $\mathcal{B}$ such that $\mathcal{O} = \mathcal{O}_{\mathcal{B}}$ can be explicitly constructed from $\mathcal{O}$.*

An algebraic characterisation

Definition 4 *An n-ary operation f is a 1-tolerant polymorphism of R if, for any tuples $\overline{a}_1, \ldots, \overline{a}_n$ at least $n-1$ of which belong to R, the tuple obtained by applying f componentwise also belongs to R.*

Theorem 15 (Larose,Loten,Tardif'06) *A structure $\mathcal{B}$ has finite duality iff its core has a 1-tolerant NU polymorphism.*

Domination, square, and diagonal

Definition 5 *An element a is said to dominate an element b in a structure $\mathcal{B}$ if, in any tuple in any relation R in $\mathcal{B}$, any number of occurrences of b can be replaced with a without leaving R.*

Example: if $(b, c, b) \in R$ then (a, c, b), (b, c, a), and (a, c, a) are all in R.

Definition 6 *The square $\mathcal{B}^2$ of a structure $\mathcal{B}$ has the base set B^2 and, for any relation $R^{\mathcal{B}}$, $R^{\mathcal{B}^2}$ contains all tuples $((a_1, b_1), \ldots, (a_n, b_n))$ with $\overline{a}, \overline{b} \in R^{\mathcal{B}}$.*

The diagonal of the square $\mathcal{B}^2$ is the substructure of $\mathcal{B}^2$ induced by $\{(b, b) \mid b \in B\}$ (NB. It is isomorphic to $\mathcal{B}$).

A combinatorial characterisation

Definition 7 *A structure $\mathcal{B}$ is said to dismantle to its substructure $\mathcal{C}$ if $\mathcal{C}$ can obtained from $\mathcal{B}$ by successive removing of dominated elements.*

Theorem 16 (Larose,Loten,Tardif'06) *A structure $\mathcal{B}$ has finite duality iff it has a retract whose square (greedily) dismantles to its diagonal.*

From this theorem, deciding of finite duality is in **NP**.

The complexity of meta-problem

Theorem 17 (Larose,Loten,Tardif'06)

1. *The problem of deciding whether an arbitrary structure $\mathcal{B}$ has finite duality is* **NP***-complete.*

2. *The problem of deciding whether a core structure $\mathcal{B}$ has finite duality is in* **PTIME***.*

Summary

	Finite duality	Pathwidth duality	Treewidth duality
Logic	FO	linear Datalog	Datalog
Complexity	$\mathbf{AC}^0$	**NL**	**PTIME**
Hierarchy	N/A	does not collapse	???
Suff. algebr. conditions	1-tolerant NU polym. for core	majority polymorphism	NU, 2-semilattice some more
Classif. in special cases		only for $\|B\| = 2$	for $\|B\| \leq 3$ or with all unary relations
Full classif.	yes	???	conjecture
Meta-problem	general – **NP**-c, cores – **PTIME**	$(1, k)$-PWD decidable	$(1, k)$-TWD dec. 1-TWD **NP**-hard

Some open problems

1. Show that a structure $\mathcal{B}$ has BTW duality whenever the corresponding algebra $\mathbf{A}_{\mathcal{B}}$ generates a CD variety. (First step is made by Kiss and Valeriote).
2. Prove (or disprove) the conjecture about BTW duality.
3. Is there $\mathcal{P}_l(\mathcal{B})$ such that $\mathcal{P}_l(\mathcal{B}) \to \mathcal{B}$ iff $\mathcal{B}$ has l-TWD?
4. Is there CSP($\mathcal{B}$) in **NL** such that $\mathcal{B}$ has no BPW?
5. Find other sufficient algebraic conditions for a structure to have BPW duality. NU polymorphism?
6. Can FO-definable CSP($\mathcal{B}$) always be defined by a short FO-formula?