# Exponential Lower Bound for tree-like Lovász-Schrijver proof systems

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Complexity of Constraints

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# Proof Systems

## Definition ([Cook and Reckhow, 1979])

A *proof system* for a language $L$ is a polynomial-time computable function mapping strings in some finite alphabet (proof candidates) onto $L$ (whose elements are considered as theorems).

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Let TAUT denote the co-NP-complete language of tautologies in DNF (equivalently, UNSAT is language of unsatisfiable formulas in CNF). A *propositional proof system* is proof system for TAUT (UNSAT).

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# Proof System Complexity

Motivation:

- NP$\neq$co-NP (and therefore P$\neq$NP) if and only if no short, easily-verifiable proofs of some tautology exist [Cook and Reckhow, 1979].
- Proof system is the extension of SAT algorithm. If $A$ is SAT algorithm then proof of $\phi$ is the protocol of $A(\phi)$.
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# Semi-algebraic Proof Systems

$$\bigvee x_i \vee \bigvee \neg y_i \quad \rightarrow \quad \sum x_i + \sum (1 - y_i) - 1 \geq 0$$

$$\frac{}{x^2 - x \geq 0} \qquad \frac{}{x \geq 0} \qquad \frac{}{1 - x \geq 0} \qquad \frac{f \geq 0 \qquad g \geq 0}{\alpha f + \beta g \geq 0}$$

Gomory-Chvátal (CP)

$$\frac{\sum c a_i x_i \geq A}{\sum a_i x_i \geq \lceil A/c \rceil}$$

$c$, $a_i$ and $A$ are integers

Lovász-Schrijver (LS)

$$\frac{h \geq 0}{hx \geq 0} \qquad \frac{h \geq 0}{h(1 - x) \geq 0}$$

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# Related Work

## Lower Bounds

- Starting from lower bound for Resolution [Tseitin, 1968]
- Exponential size lower bound for CP on Clique-Coloring Tautologies [Pudlák, 1997]
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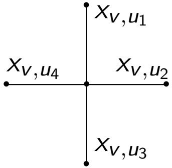
## Our Result (Informally)

We prove exponential lower bound for tree-like LS as propositional proof system.

# Tseitin formulas

$G = (V, E)$, to each $e \in E$ attach $\{0,1\}$-variable $x_e$, fix $V' \subseteq V$, $|V'|$ is odd,




$$T_G = \begin{cases} \oplus_{v \ni e} y_e = 1, & \text{for all } x \in V' \\ \oplus_{v \ni e} y_e = 0, & \text{for all } x \in V \setminus V' \end{cases}$$

$\sum x_{v,u_i} = 1 \mod 2$

$T_G$ is contradiction.

$G = (V, E)$ is $(r, d, c)$-expander iff degrees of $v \in V$ are less than $d$ and for any $A \subset V, |A| \leq r$, $|\partial(A)| \geq c|A|$, where boundary $\partial(A) = \{(v, w) | v \in A, w \in V \setminus A\}$.

Example:




$\partial(v) = \{(v, w), (v, u)\}$
$(1, 2, 2)$-expander

Lemma ([Alekhnovich et al., 2004])

*If $G$ is $(r, d, c)$-expander and we remove not more than $cr/4$ edges from it, then we can obtain a $(r/2, d, c/2)$-expander by removing some other edges and vertices.*

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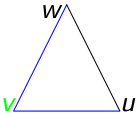


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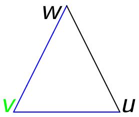


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Static $\mathrm{LS}_+$ [Grigoriev et al., 2002]

$$\sum_{i=1}^{M} f_i \sum_{l} g_{i,l} = -1 \ ,$$

where $g_{i,l} = c_{i,l} \cdot \prod_{k \in U_{i,l}^+} x_k \cdot \prod_{k \in U_{i,l}^-} (1 - x_k)$, coefficients $c_{i,l} \in \mathbb{R}^+$ and $f_i$ are axioms

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# Our result

**Theorem**

*Any tree-like $LS_+$ refutation of Tseitin formula $T_G$ for a connected $d$-regular $(r = n/2, d, c)$-expander $G$ with $n$ vertices and $c > 2$ has size $exp(\Omega(n))$.*

# Proof Sketch

- Prove that every static $LS_+$ proof of Tsejtin formula based on enough good expander contains at least one polynomial with special property (in fact: polynomials with big enough number of variables).
- Make linear number of substitution of variables such a way that:
  1) all exept may by exponential small fraction of polynomials with special property will be removed from the proof;
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## First step: polynomial with special property

- Existence polynomials with special property in all *Positivstellensatz* proofs of Tseitin formulas on expanders [Grigoriev, 2001]

$$\bigvee x_i \vee \bigvee \neg y_i \quad \rightarrow \quad \prod (1 - x_i) \cdot \prod y_i = 0$$

axioms $x^2 - x = 0$, proof is the set of polynomials $g_1, \ldots, g_{m+n}$ and $h_1, \ldots, h_l$ such that

$$\sum_{i=1}^{m+n} f_i g_i = 1 + \sum_{j=1}^{l} h_j^2 \ .$$

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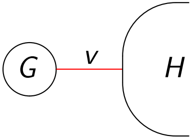


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## Open Questions

- Prove lower bound for DAG-like LS
- Prove lower bound for CP on Tsejtin formulas

Thank you!

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