Constraint Satisfaction with Succinctly Specified Relations

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Joint work with Hubie Chen
CSP instances

\[ \mathcal{I} = (V, D, C) \]

- \( V \): variables
- \( D \): domain
- \( C \): constraints of the form \( Rx_1 \ldots x_k \)
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Two relational structures associated with \( \mathcal{I} \):

- \( A(\mathcal{I}) \) structure induced by constraints on the variables.
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- \( B(\mathcal{I}) \) structure induced by constraint relations on the domain.
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variables, domain, constraints of the form \( Rx_1 \ldots x_k \)

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- \( \mathcal{B}(\mathcal{I}) \) structure induced by constraint relations on the domain. Restrictions on \( \mathcal{B}(\mathcal{I}) \) are called constraint language restrictions.
Uniform CSPs

For classes $K, L$ of relational structures:

**CSP($K, L$)**

**Instance:** CSP instance $\mathcal{I}$

with $A(\mathcal{I}) \in K$ and $B(\mathcal{I}) \in L$.

**Problem:** Decide if $\mathcal{I}$ has a solution.
Uniform CSPs

For classes $\mathbf{K}, \mathbf{L}$ of relational structures:

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**Instance:** CSP instance $\mathcal{I}$
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**Problem:** Decide if $\mathcal{I}$ has a solution.

**Remark**

- We write CSP($\mathbf{K}$, $-$) or CSP($-$, $\mathbf{L}$) if $\mathbf{L}$ resp. $\mathbf{K}$ is the class of all structures.
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Remark

- We write $\text{CSP}(K, -)$ or $\text{CSP}(-, L)$ if $L$ resp. $K$ is the class of all structures.
- We are mainly interested in restrictions of the form $\text{CSP}(K, -)$ (structural restrictions).
Tractable structural restrictions

Two important ideas

1. Instances with a tree-like structure can be solved in polynomial time by dynamic programming.
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Definition
A class of $K$ structures has bounded tree width modulo homomorphic equivalence if there is a $k$ such that every structure in $K$ is homomorphically equivalent to a structure of tree width at most $k$. 

Theorem (Dalmau, Kolaitis, and Vardi 2002)
Let $K$ be a class of structures of bounded tree width modulo homomorphic equivalence. Then $CSP(K, -)$ is tractable.
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Bounded arity

The **arity** of a structure is the maximum of the arities of its relations.
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Theorem (G. 2003)
Assume that $\text{FPT} \neq \text{W}[1]$ (or stronger: $\text{3-SAT} \notin \text{DTIME}(2^{o(n)})$).

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**Theorem (G. 2003)**

Assume that \( \text{FPT} \neq \text{W}[1] \) (or stronger: \( \text{3-SAT} \not\in \text{DTIME}(2^{o(n)}) \)). For every recursively enumerable class \( K \) of structures of bounded arity, the following are equivalent:

1. \( \text{CSP}(K, -) \) is tractable.
2. \( K \) has bounded tree width modulo homomorphic equivalence.
Digression — a few remarks on the theorem

Theorem
Assume that $\text{FPT} \neq \text{W}[1]$ (or stronger: $\text{3-SAT} \not\in \text{DTIME}(2^{o(n)})$).
For every recursively enumerable class $\mathbf{K}$ of structures of bounded arity, the following are equivalent:
1. $\text{CSP}(\mathbf{K}, -)$ is tractable.
2. $\mathbf{K}$ has bounded tree width modulo homomorphic equivalence.

▶ This is not a dichotomy theorem.
▶ The assumption $\text{FPT} \neq \text{W}[1]$ cannot be replaced by anything weaker.
▶ The requirement that $\mathbf{K}$ be recursively enumerable is inessential.
▶ The requirement that $\mathbf{K}$ be of bounded arity is crucial!

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Assume that $\text{FPT} \neq \text{W}[1]$ (or stronger: $3\text{-SAT} \not\in \text{DTIME}(2^{o(n)})$). For every recursively enumerable class $K$ of structures of bounded arity, the following are equivalent:
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Assume that FPT \(\neq W[1]\) (or stronger: 3-SAT \(\not\in\) DTIME\((2^{o(n)})\)). For every recursively enumerable class \(K\) of structures of bounded arity, the following are equivalent:
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There are classes \(K\) of graphs of unbounded treewidth modulo homomorphic equivalence such that CSP\((K, -)\in NP[\log^2 n]\).
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- The assumption $\text{FPT} \neq \text{W}[1]$ cannot be replaced by anything weaker.
  If $\text{FPT} = \text{W}[1]$ then there are classes $K$ of graphs of unbounded treewidth modulo homomorphic equivalence such that $\text{CSP}(K, -)$ is tractable.

▶ The requirement that $K$ be recursively enumerable is inessential.
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▶ The assumption $\text{FPT} \neq \text{W}[1]$ cannot be replaced by anything weaker.
▶ The requirement that $K$ be recursively enumerable is inessential. With a slightly stronger complexity theoretic assumption, we can get rid of it.
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Theorem
Assume that FPT ≠ W[1] (or stronger: 3-SAT ∉ DTIME(2^{o(n)})). For every recursively enumerable class \( \mathcal{K} \) of structures of bounded arity, the following are equivalent:
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If all variables are covered by a bounded number of constraints, then the number of solutions is polynomially bounded.
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Let $k$ be a constant and $\mathcal{I} = (V, D, C)$ a CSP-instance. Suppose that the constraints

$$R_1 x_{11} \ldots x_{1r_1}, \ldots, R_k x_{k1} \ldots x_{kr_k}$$

cover all variables, i.e., $V \subseteq \{x_{ij} \mid 1 \leq i \leq k, 1 \leq j \leq r_i\}$. 
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cover all variables, i.e., $V \subseteq \{x_{ij} \mid 1 \leq i \leq k, 1 \leq j \leq r_i\}$. Then for the set $S$ of all solutions of $\mathcal{I}$, we have:

$$S \subseteq R_1 \Join \cdots \Join R_k$$

($S \subseteq R_1 \times \cdots \times R_k$ if the variable sets are disjoint).
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($S \subseteq R_1 \times \cdots \times R_k$ if the variable sets are disjoint). Hence

$$|S| \leq |R_1| \cdot |R_2| \cdots |R_k| \leq |\mathcal{I}|^k,$$

and $S$ can be computed in polynomial time.
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Variants and extensions

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- These ideas can be combined with tree decompositions, leading to hypertree width and fractional hypertree width.

That’s all great — but wait …
haven’t we just proved
\((N = 1 \lor P = 0)\)

If viewed as a (Boolean) CSP, the satisfiability problem \(\text{SAT}(\Phi)\) for the following class of formulas is tractable:

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\Phi := \{ \phi \land (x_1 \lor \ldots \lor x_n) \mid n \geq 1, \phi \text{ CNF formula with variables among } x_1, \ldots, x_n \}
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- Hence \(P = \text{NP}\).
By a simple reduction to constraint satisfaction problems, we have proved the satisfiability problem to be in polynomial time and hence \( P = NP \).

This impressively shows the power of the algorithmic ideas developed in the area.
Explicit and implicit representations

- We usually assume the constraint relations to be specified explicitly, that is, by listing all tuples.

As long as the arity is bounded (in particular for finite constraint languages), this is irrelevant, because the size of the relations is polynomially bounded in terms of the domain. If the arity is unbounded, the size of the relations may be exponential in the domain size, which affects the complexity. For problems of unbounded arity that occur in practice, e.g., SAT, LINEAR PROGRAMMING, constraints are usually specified implicitly, e.g. by clauses or linear inequalities. This makes the problems harder!

In this work, we try to initiate a systematic study of the complexity of CSP with implicitly represented relations.
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The Boolean case

Obvious idea
Specify constraints by Boolean circuits or formulas.
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Problem
Even structurally very simple instances become hard:
The Boolean CSP with one $n$-ary constraint

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where $R$ is represented by a Boolean circuit, is NP-complete.
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DNF Representations
DNF-formulas form natural large class of formulas with a tractable satisfiability problem.
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DNF Representations
DNF-formulas form natural large class of formulas with a tractable satisfiability problem.
Therefore, we decided to choose DNF-formulas as succinct representations of Boolean constraint relations.
Arbitrary domains — the GDNF-representation

Definition
Let $D$ be a set. A generalised DNF (GDNF) representation of a relation $R \subseteq D^k$ is an expression of the form

$$\bigcup_{i=1}^{m} (P_{i1} \times \cdots \times P_{ik})$$

where $m \geq 0$ and $P_{ij} \subseteq D$ for $1 \leq i \leq m, 1 \leq j \leq k$. 

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where $m \geq 0$ and $P_{ij} \subseteq D$ for $1 \leq i \leq m, 1 \leq j \leq k$.

Remark
The GDNF enables us to represent relations of size $\Omega(D^k)$ by expressions of size $O(m \cdot |D| \cdot k)$. 
Example

Notation
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Let $\text{MAX} \subseteq [n]^{k+1}$ be defined by

$$\text{MAX} = \{(x_1, \ldots, x_k, y) \mid y = \max\{x_1, \ldots, x_k\}\}$$

Then $|\text{MAX}| = \Omega(n^k)$.
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Then \( |\text{MAX}| = \Omega(n^k) \).

\( \text{MAX} \) has the following GDNF-representation of size \( O(k^2 \cdot n^2) \):
\[
\bigcup_{1 \leq i \leq k} \big( [j] \times \ldots \times [j] \times \{i\} \times [j] \times \ldots \times [j] \times \{i\} \big).
\]
Definition

- A CSP-instance $\mathcal{I}$ is represented in GDNF if all constraint relations are represented in GDNF.
Succinctly specified CSP

Definition

- A CSP-instance $\mathcal{I}$ is represented in GDNF if all constraint relations are represented in GDNF.
- By $\text{CSP}_{\text{GDNF}}(K, L)$ is the version of the problem $\text{CSP}(K, L)$ where instances are represented in GDNF.
The incidence structure

Definition

1. Incidence vocabulary of a relational vocabulary \( \tau \):

\[
\tau_I = \{ R_1, \ldots, R_k \mid R \in \tau \text{ } k\text{-ary} \}
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\((R_i \text{ binary})\).
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1. **Incidence vocabulary** of a relational vocabulary \( \tau \):

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2. **Incidence structure** of a \( \tau \)-structure \( \mathcal{A} \): \( \tau_I \)-structure \( \mathcal{A}_I \) with
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\[
\mathcal{A}_I = \mathcal{A} \cup \bigcup_{R \in \sigma} \{(R, a_1, \ldots, a_k) : (a_1, \ldots, a_k) \in R^\mathcal{A}\},
\]
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1. **Incidence vocabulary** of a relational vocabulary \( \tau \):
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2. **Incidence structure** of a \( \tau \)-structure \( A \): \( \tau_I \)-structure \( A_I \) with
   
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     \[
     A_I = A \cup \bigcup_{R \in \sigma} \{(R, a_1, \ldots, a_k) : (a_1, \ldots, a_k) \in R^A\},
     \]
   
   - relations
     \[
     R^A_i = \{((R, a_1, \ldots, a_k), a_i) : (a_1, \ldots, a_k) \in R^A\}
     \]
     for all \( k \)-ary \( R \in \tau \) and \( i \in [k] \).
\[ \{ R, E, P \} \text{- structure } \mathcal{A} : \]

\[ \mathcal{A} = \{ 1, 2, 3, 4 \} \]

\[ R^+= \{ (1, 2, 3), (2, 3, 4) \} \]

\[ E^+= \{ (1, 2), (2, 3), (3, 4), (4, 1) \} \]

\[ P^+= \{ (1) \} \]
Definition
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Remark
For all structures $\mathcal{A}$ of arity $k$:

$$iw(\mathcal{A}) - 1 \leq tw(\mathcal{A}) \leq (iw(\mathcal{A}) + 1) \cdot (k - 1).$$
Theorem
Assume that $\text{FPT} \not= \text{W}[1]$.
For every recursively enumerable class $\mathbf{K}$ of structures of bounded arity, the following are equivalent:

1. $\text{CSP}_{\text{GDNF}}(\mathbf{K}, -)$ is tractable.
2. $\mathbf{K}$ has bounded incidence width modulo homomorphomic equivalence.
Theorem

Let $L_k$ be the class of all relational structures having a near-unanimity polymorphism of arity $k$. For each $k \geq 3$, the problem $\text{CSP}_{\text{GDNF}}(\neg, L_k)$ is tractable.
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Let $L$ be the set of all relational structures invariant under a set function. The problem $\text{CSP}_{\text{GDNF}}(-, L)$ is tractable.
Ordered binary decision diagrams (OBDDs) are another representation of Boolean relations with nice algorithmic properties. In particular, the satisfiability problem is decidable.
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We considered the natural generalization of OBDDs to arbitrary finite domains and obtained initial results.