Definability of CSPs in Fixed-Point and Infinitary Logics

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Some Preliminaries

We consider non-uniform CSP, i.e. problems of the form CSP$(\mathbb{B})$ for a finite relational structure $\mathbb{B}$.

We say CSP$(\mathbb{B})$ is definable in a logic L if there is a formula φ of L such that

$$\mathbb{A} \models \varphi \quad \text{if, and only if,} \quad \mathbb{A} \longrightarrow \mathbb{B}.$$

We write CSP for the class of all problems of the form CSP$(\mathbb{B})$.

We write $\overline{\text{CSP}(\mathbb{B})}$ for the complement of CSP$(\mathbb{B})$ and co-CSP for the class of problems that are complements of problems in CSP.

Datalog definable CSPs

There are many equivalent characterisations of the CSPs whose complements are definable in Datalog.

For any finite relational structure $\mathbb{B}$, the following are equivalent:

- CSP$(\mathbb{B})$ has finite width.
- $\overline{\text{CSP}(\mathbb{B})}$ is definable in Datalog.
- $\overline{\text{CSP}(\mathbb{B})}$ is definable in $\exists L^{\omega,+}_{\infty,\omega}$.
- For some k, the existential k-pebble game determines membership in CSP$(\mathbb{B})$.
- CSP$(\mathbb{B})$ has bounded treewidth duality.

First-Order definable CSPs

The following are equivalent:

- $\mathrm{CSP}(\mathbb{B})$ is definable in FO.
- $\overline{\mathrm{CSP}(\mathbb{B})}$ is definable in existential positive FO.
- $\mathrm{CSP}(\mathbb{B})$ has finite duality.

(Atserias; Rossman)

Fixed-point and Infinitary Logics

A map of some logics studied in descriptive complexity, with increasing expressive power to the right and up.

	LFP + Count	$C^{\omega}_{\infty\omega}$
FO	LFP	$L^{\omega}_{\infty\omega}$
$\exists$FO^{+}	Datalog	$\exists L^{\omega,+}_{\infty,\omega}$

For co-CSP, FO $=$ $\exists$FO^{+} and Datalog $= \exists L^{\omega,+}_{\infty,\omega}$.

- Is every CSP$(\mathbb{B})$ in LFP of finite width?
- Is every CSP$(\mathbb{B})$ in $L^{\omega}_{\infty\omega}$ also in LFP?
- Is every CSP$(\mathbb{B})$ in $L^{\omega}_{\infty\omega}$ of finite width?

Fixed-point and Infinitary Logics

Datalog is (*essentially*) the extension of $\exists$FO$^+$ with recursive definitions:

$$R(\mathbf{x}) \leftarrow \varphi(R) \qquad \text{where } \varphi \text{ is existential positive.}$$

LFP is (*essentially*) the extension of FO with recursive definitions:

$$R(\mathbf{x}) \leftarrow \varphi(R) \qquad \text{where } \varphi \text{ is positive in the symbol } R.$$

$L^\omega_{\infty\omega}$ is the extension of FO with infinitary conjunctions and disjunctions but allowing only finitely many variables in each formula.

LFP $+$ Count and $C^\omega_{\infty\omega}$ are the extensions by means of a counting mechanism of LFP and $L^\omega_{\infty\omega}$ respectively.

non-Datalog definable CSPs

Two canonical CSPs that are not Datalog definable.

3-colourability, CSP(K_3)

- NP-complete
- not Datalog definable
- all polymorphisms of K_3 essentially unary
- not in $C^{\omega}_{\infty\omega}$ **(D. 1998)**

Linear equations over $\mathbb{Z}_2$

- tractable by Gaussian elimination
- not Datalog definable **(Feder-Vardi 1998)**
- a Mal'tsev polymorphism
- in LFP or LFP $+$ Count?

Systems of Equations

Let $(\mathbb{Z}_m)_r$ denote the finite structure with universe $\{0, \ldots, m-1\}$ and relations R_i^j ($i \leq m, j \leq r$):

$$R_i^j(a_1, \ldots, a_j) \quad \text{if, and only if,} \quad a_1 + \cdots + a_j \equiv i \pmod{m}.$$

CSP$((\mathbb{Z}_m)_r)$ codes solvable systems of equations over the group $(\mathbb{Z}_m, +)$ (with at most r variables per equation).

Theorem:

CSP$((\mathbb{Z}_m)_r)$ is not definable in $C^{\omega}_{\infty\omega}$ for any $m \geq 2$, $r \geq 3$.

Construction

The construction for $(\mathbb{Z}_2)_3$ is simpler.

Take $\mathcal{G}$ a 3-regular, connected graph with treewidth $> k$.

Define equations $\mathbf{E}(\mathcal{G})$ with two variables x_e^0, x_e^1 for each edge.

For each vertex v with edges e_1, e_2, e_3 incident on it, we have eight equations:

$$E_v : \qquad x_{e_1}^i + x_{e_2}^j + x_{e_3}^k = i + j + k \pmod 2$$

The system of equations $\tilde{\mathbf{E}}(\mathcal{G})$ is obtained from $\mathbf{E}(\mathcal{G})$ by replacing, for exactly one vertex v, E_v by:

$$E_v' : \qquad x_{e_1}^i + x_{e_2}^j + x_{e_3}^k = i + j + k + 1 \pmod 2$$

Facts about the Construction

Fact: $\mathbf{E}(\mathcal{G}) \equiv^{C^k_{\infty\omega}} \tilde{\mathbf{E}}(\mathcal{G})$

by a bijective game argument.

Fact: $\mathbf{E}(\mathcal{G})$ is satisfiable.

by setting the variables x_e^i to i.

Fact: $\tilde{\mathbf{E}}(\mathcal{G})$ is unsatisfiable.

by a counting argument.

A somewhat more involved construction is required for $\mathbb{Z}_m$, $m > 2$.

Varieties and Reductions

For a finite structure $\mathbb{B}$, write $\mathcal{B}$ for its algebra of polymorphisms and $\mathrm{var}(\mathcal{B})$ for the variety that $\mathcal{B}$ generates.

Theorem (Bulatov, Jeavons, Krokhin)
CSP$(\mathbb{B})$ is tractable if, and only if, for every $\mathbb{B}'$ with $\mathcal{B}' \in \mathrm{var}(\mathcal{B})$, CSP$(\mathbb{B}')$ is tractable.

Theorem (Larose, Zadori)
CSP$(\mathbb{B})$ is finite width if, and only if, for every $\mathbb{B}'$ with $\mathcal{B}' \in \mathrm{var}(\mathcal{B})$, CSP$(\mathbb{B}')$ is finite width.

Both are proved by showing reductions corresponding to subalgebras, homomorphic images and powers.

Reductions

Let $\mathbb{B}$ and $\mathbb{B}'$ be *non-degenerate* structures:

Lemma:

- If $\mathcal{B}' = \mathcal{B}^n$ for some n, then CSP$(\mathbb{B}') \leq_{\mathrm{qf}}$ CSP$(\mathbb{B})$.
- If $\mathcal{B}' \subseteq \mathcal{B}$, then CSP$(\mathbb{B}') \leq_{\mathrm{qf}}$ CSP$(\mathbb{B})$.
- If $\mathcal{B}' = h(\mathcal{B})$ for some homomorphism h, then CSP$(\mathbb{B}') \leq_{\mathrm{qf}}$ CSP$(\mathbb{B})$.

Here, $\leq_{\mathrm{qf}}$ denotes *quantifier-free reductions*. These are very weak reductions preserving definability in all logics we are considering.

Reduction to the Idempotent Case

For a finite structure $\mathbb{B}$, let $\mathbb{D}$ be its expansion with a singleton unary relation R_b for each element b of $\mathbb{B}$.

Then, the polymorphisms of $\mathbb{D}$ are exactly the idempotent polymorphisms of $\mathbb{B}$.

We show:

- CSP$(\mathbb{B}) \leq_{\mathrm{qf}}$ CSP$(\mathbb{D})$
- if $\mathbb{B}$ is a core with at least two elements then CSP$(\mathbb{D}) \leq_{\mathrm{ep}}$ CSP$(\mathbb{B})$

Here $\leq_{\mathrm{ep}}$ denotes a reduction defined by an *existential positive* formula.

We can similarly reduce to the non-degenerate case, but this requires $\leq_{\text{Datalog}}$.

Hobby-McKenzie Types

Let $\mathbb{B}$ be a structure with idempotent algebra of polymorphisms $\mathcal{B}$.

Recall (from Matt Valeriote's talk), the types (*unary, affine*) of neighbourhoods.

Theorem (Bulatov, Jeavons, Krokhin)

If $\mathrm{var}(\mathcal{B})$ admits the unary type, then CSP$(\mathbb{B})$ is NP-complete.

Conjecture: Otherwise CSP$(\mathbb{B})$ is tractable.

Theorem (Larose, Zadori)

If $\mathrm{var}(\mathcal{B})$ admits the unary or affine types, then CSP$(\mathbb{B})$ is not of finite width.

Conjecture: Otherwise $\overline{\text{CSP}(\mathbb{B})}$ is Datalog.

Some Consequences

Theorem

Let $\mathbb{B}$ be non-degenerate with idempotent $\mathcal{B}$

- If $\mathrm{var}(\mathcal{B})$ admits the unary type, then CSP$(K_3) \leq_{\mathrm{qf}}$ CSP$(\mathbb{B})$.
- If $\mathrm{var}(\mathcal{B})$ admits the affine type, then CSP$((\mathbb{Z}_m)_r) \leq_{\mathrm{qf}}$ CSP$(\mathbb{B})$ for some m and r.

We can relax the conditions on $\mathbb{B}$ by relaxing $\leq_{\mathrm{qf}}$ accordingly to $\leq_{\text{FO}}$ or $\leq$ Datalog.

Some Consequences

Corollary

If CSP$(\mathbb{B})$ is definable in $C^{\omega}_{\infty\omega}$ then $\mathrm{var}(\mathcal{B})$ omits the unary and affine types.

This may be seen as a strengthening of the Larose-Zadori theorem, in that we have replaced Datalog by $C^{\omega}_{\infty\omega}$.

But, is it really stronger? Are there problems in co-CSP in $C^{\omega}_{\infty\omega}$ that are not in Datalog?

If the *Larose-Zadori finite width conjecture* is true, then the above corollary shows there are none.

LFP vs. Datalog

(work in progress with Stephan Kreutzer)

There are classes of finite structures, definable in LFP that are closed under homomorphisms but not definable in Datalog.

this answers a question posed by Atserias.

Let P be the class of directed graphs with distinguished s and t such that P either contains a cycle or a path from s to t whose length is a square.

For the proof that P is not Datalog, we use the "pumping lemma" of Afrati, Cosmadakis and Yannakakis.

$\bar{P}$ is of the form CSP$(\mathbb{B})$ for an *infinite* $\mathbb{B}$.