Some other Galois connections

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$A : \text{fixed finite set, } |A| \geq 2$

Galois connection $\text{Inv} – \text{Pol}$

between $\mathcal{P} \mathbf{O}_A$ and $\mathcal{P} \mathbf{R}_A$

based on “$f \text{pres} \varrho$”

$\text{Pol Inv } F = \text{clone}(F)$

$\text{Inv Pol } \Gamma = \langle \Gamma \rangle_{\exists, \land, =, \text{false}}$

From this connection we can obtain other Galois connections, also based on the relationship $\text{pres}$, by restricting or extending the sets $\mathbf{O}_A$ or $\mathbf{R}_A$. 
Example

Let $sO_A := \{ f \in O_A \mid f \text{ surjective} \}$

For $\Gamma \subseteq R_A$ put $sPol \Gamma := sO_A \cap Pol \Gamma$.

Then $\text{Inv} - sPol$ is a Galois connection between $\mathcal{P}sO_A$ and $\mathcal{P}R_A$

$sPol \text{Inv} F = sO_A \cap \text{clone}(F)$

$= \text{surjective part of a clone}$

_**better:** $sPol \Gamma \mapsto \text{clone}(sPol \Gamma) = Pol \text{Inv} sPol \Gamma$

A clone $C$ is _surjectively generated_:

$\iff C = \text{clone}(sO_A \cap C')$

$(sO_A \cap Pol \text{Inv} sPol \Gamma = sPol \Gamma,$

$\text{Inv}(Pol \text{Inv} sPol \Gamma) = \text{Inv} sPol \Gamma$)

$\text{Inv} sPol \Gamma = \langle \Gamma \rangle_{\forall, \exists, \wedge, =}$
The Quantified CSP

Let $\Gamma \subseteq R_A$ be a finite set of relations. Then $QCSP(\Gamma)$ denotes the following decision problem:

**Input:** A sentence of the form

$$(Q_1x_1) \cdots (Q_nx_n) (\varrho_1(x_{i_1,1}, \ldots) \land \cdots \land \varrho_m(x_{i_m,1}, \ldots))$$

with $\varrho_j \in \Gamma$, $i_{s,t} \in \{1, \ldots, n\}$, $Q_j \in \{\forall, \exists\}$.

**Problem:** Is this sentence true?
Assume $\Gamma_2 \subseteq \langle \Gamma_1 \rangle_{\forall, \exists, \land, =}$.

Then for every relation $\sigma \in \Gamma_2$ there is a formula $\varphi \in \Phi(\forall, \exists, \land, =)$ with

$$\sigma(x_1, \ldots, x_m) \iff \varphi(\rho_1, \ldots, \rho_l; x_1, \ldots, x_m),$$

where $\rho_1, \ldots, \rho_l \in \Gamma_1$. If we have an input of $\text{QCSP}(\Gamma_2)$, then the $\sigma \in \Gamma_2$ can be replaced by the formulas on the right side. Then the new sentence can be transformed into a normal form that corresponds to an instance of $\text{QCSP}(\Gamma_1)$.

It can be shown that this is possible in polynomial time.

$$\Gamma_2 \subseteq \langle \Gamma_1 \rangle_{\forall, \exists, \land, =} \iff \Gamma_2 \subseteq \text{Inv sPol} \Gamma_1$$
$$\iff \text{sPol} \Gamma_1 \subseteq \text{sPol} \Gamma_2$$

**Theorem.** Let $\Gamma_1, \Gamma_2 \subseteq \mathbf{R}_A$ be finite sets of relations. If $\text{sPol} \Gamma_1 \subseteq \text{sPol} \Gamma_2$, then $\text{QCSP}(\Gamma_2)$ is polynomial-time reducible to $\text{QCSP}(\Gamma_1)$.

If $\text{sPol} \Gamma_1 = \text{sPol} \Gamma_2$, then $\text{QCSP}(\Gamma_1)$ and $\text{QCSP}(\Gamma_2)$ are polynomial-time equivalent.
### Restricting $O_A$

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$$\text{LOC}_n \Gamma := \{ \varrho \in R_A \mid (\forall \varrho_0 \subseteq \varrho) \\
(|\varrho_0| \leq n \Rightarrow (\exists \sigma \in \Gamma) (\varrho_0 \subseteq \sigma \subseteq \varrho) \}$$
Extending $O_A$

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$\text{pHyp}_A := \{ f \mid f : A^n \to \mathcal{P} A, \ n \in \mathbb{N}^+ \}$

$\text{Hyp}_A := \{ f \mid f : A^n \to \mathcal{P} A \setminus \{\emptyset\}, \ n \in \mathbb{N}^+ \}$

$\text{Par}_A :=$ set of all partial functions on $A$
Recipes

Starting from partial hyperfunctions, there are roughly the following connections between elements of the closure on the relational side and properties of the partial hyperfunctions, used in the Galois connections.

\( \land, \text{false} \) (all partial hyperfunctions)
\( = \) unique, \(|f(\bar{a})| \leq 1\)
\( \neq \) injective
\( \lor \) unary
\( \exists \) everywhere defined, \(|f(\bar{a})| \geq 1\)
\( \forall \) surjective
\( \neg \) ivolutable, invertable (strong invariance)

Remarks:

\[ \langle \Gamma \rangle \forall, \exists, \land, \lor, = \] = \[ \langle \Gamma \rangle \forall, \exists, \land, \lor, =, \neq (|A| \geq 2) \]
\[ \langle \Gamma \rangle \exists, \land, =, \neq \] = \[ \langle \Gamma \rangle \forall, \exists, \land, \lor, =, \neq \text{ if } |A| \geq 3 \]
Main idea for the proof

The difficult part of the proof is:

\[ \Gamma_0 \text{ relational clone } \Rightarrow \text{Inv Pol}|\Gamma_0 \subseteq \Gamma_0 \]

So let \( \varrho \in \text{Inv Pol}|\Gamma_0 \) with arity \( m \). We want to show \( \varrho \in \Gamma_0 \).

If \( \varrho = \emptyset \), then \( \varrho \in \Gamma_0 \), because a relational clone contains all empty relations. Otherwise let \( \varrho = \{ \vec{r}_1, \ldots, \vec{r}_n \} \) with \( \vec{r}_j \in A^m \).

Consider \( \mu_{\Gamma_0} (\varrho) := \bigcap \{ \sigma \mid \sigma \in \Gamma_0 \text{ and } \varrho \subseteq \sigma \} \)

\( \Gamma_0 \) is \( \cap \)-closed, therefore

\[ \varrho \in \Gamma_0 \iff \varrho = \mu_{\Gamma_0} (\varrho) \]

Let \( \vec{b} \in \mu_{\Gamma_0} (\varrho) \) We have to show \( \vec{b} \in \varrho \). Then \( \varrho = \mu_{\Gamma_0} (\varrho) \) and the proof is finished.

Consider the partial assignment \( (\vec{r}_1, \ldots, \vec{r}_n) \mapsto \vec{b} \), i.e.

\( (\vec{r}_1(i), \ldots, \vec{r}_n(i)) \mapsto \vec{b}(i) \) for all \( i \) with \( 1 \leq i \leq m \). The main idea in this proof, and similar also the main idea for the proofs for the other Galois connections, is the following:

**Extend this partial assignment to a function \( g \) in Pol|\Gamma_0 \)**

If this is possible, then \( \varrho \in \text{Inv Pol}|\Gamma_0 \) implies that \( g \) preserves \( \varrho \), and therefore \( \vec{b} = g(\vec{r}_1, \ldots, \vec{r}_n) \in \varrho \).
Theorem. Let $C \subseteq O_A$. Then $C$ is a clone of functions iff $C = \text{Pol } \Gamma$ for some set $\Gamma \subseteq R_A$.

If $F \subseteq O_A$, then clone $F = \text{Pol } \text{Inv } F$.

Let $\Gamma_0 \subseteq R_A$. Then $\Gamma_0$ is a relational clone iff $\Gamma_0 = \text{Inv } F$ for some set $F \subseteq O_A$.

If $\Gamma \subseteq R_A$, then $\langle \Gamma \rangle = \text{Inv } \text{Pol } \Gamma$.

What remains to prove?

$C$ clone $\iff C = \text{Pol } \text{Inv } C$

$\Gamma_0$ relational clone $\iff \Gamma_0 = \text{Inv } \text{Pol } \Gamma_0$

The “$\iff$” directions are easy.

$C \subseteq \text{Pol } \text{Inv } C$, $\Gamma_0 \subseteq \text{Inv } \text{Pol } \Gamma_0$ clear

Remains:

$C$ clone $\Rightarrow \text{Pol } \text{Inv } C \subseteq C$

$\Gamma_0$ relational clone $\Rightarrow \text{Inv } \text{Pol } \Gamma_0 \subseteq \Gamma_0$

Let $\sigma \subseteq A^m$, $\Gamma \subseteq R_A$

$\mu_\Gamma(\sigma) := \bigcap \{ \varrho \mid \varrho \in \Gamma \text{ and } \sigma \subseteq \varrho \}$

- $\sigma \subseteq \mu_\Gamma(\sigma)$
- $\Gamma \text{-closed } \Rightarrow \mu_\Gamma(\sigma) \in \Gamma$
  
  $\sigma \in \Gamma \iff \sigma = \mu_\Gamma(\sigma)$

- $\sigma_1 \subseteq \sigma_2 \Rightarrow \mu_\Gamma(\sigma_1) \subseteq \mu_\Gamma(\sigma_2)$

L1 Let $\sigma = \{ \vec{r}_1, \ldots, \vec{r}_n \}$. Then

$\mu_{\text{Inv } F}(\sigma) = \{ f(\vec{r}_1, \ldots, \vec{r}_n) \mid f \in \text{clone}^{(n)} F \}$

Proof: The right side contains $\sigma$, is in $\text{Inv } F$, and is the least invariant relation that contains $\vec{r}_1, \ldots, \vec{r}_n$. <!-- Proof: The right side contains $\sigma$, is in $\text{Inv } F$, and is the least invariant relation that contains $\vec{r}_1, \ldots, \vec{r}_n$. -->
\( \chi_n : |A|^n\)-ary relation on \( A \)

- Write down all \( n \)-tuples on \( a \) as rows of a matrix (lexicographic).
- The \( n \) columns of this matrix are the elements \( \bar{q}_1, \bar{q}_2, \ldots, \bar{q}_n \) of \( \chi_n \).
- For \( i = 1 \ldots |A|^n \), \( (\bar{q}_1(i), \bar{q}_2(i), \ldots, \bar{q}_n(i)) \) runs through all possible \( n \)-tuples.

\[
\begin{array}{ccc}
\bar{q}_1 & \bar{q}_2 & \bar{q}_3 \\
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0 \\
0 & 1 & 1 \\
1 & 0 & 0 \\
1 & 0 & 1 \\
1 & 1 & 0 \\
1 & 1 & 1 \\
\end{array}
\]

\( A = \{0, 1\}, n = 3, \chi_3 = \)

If \( \bar{c} \) is an \( |A|^n \)-tuple, then we can define an \( n \)-ary function \( g_{\bar{c}} \) by

\[ g(\bar{q}_1(i), \ldots, \bar{q}_n(i)) := \bar{c}(i) \text{ for all } i = 1, \ldots, |A|^n. \]

If \( \bar{c} \in \mu_{\text{Inv}}(\chi_n) \), then by L1, \( g_{\bar{c}} \in \text{clone}^{(n)} C = C^{(n)}. \)

Let \( f \in \text{Pol Inv} C \) with arity \( n \).

Then \( f \) preserves \( \mu_{\text{Inv}}(\chi_n) \in \text{Inv} C \). Consequently,

\[ \bar{c} := f(\bar{q}_1, \ldots, \bar{q}_n) \in \mu_{\text{Inv}} C(\chi_n) \text{ and } f = g_{\bar{c}}. \text{ Therefore } f \in C. \text{ This shows that for a clone } C \text{ always } \text{Pol Inv} C \subseteq C. \]
It remains to proof \( \text{Inv Pol} \Gamma_0 \subseteq \Gamma_0 \) for each relational clone \( \Gamma_0 \).

Let \( s : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, n\} \)
\( W_s : \mathbb{R}_A^m \to \mathbb{R}_A^n \)
\( W_s(\sigma) := \{(a_1, \ldots, a_n) \mid \sigma(a_{s(1)}, \ldots, a_{s(m)})\} \)
\( V_s : \mathbb{R}_A^n \to \mathbb{R}_A^m \)
\( V_s(\sigma) := \{(a_1, \ldots, a_m) \mid (\exists b_1) \ldots (\exists b_n) \sigma(b_1, \ldots, b_n) \land \bigwedge_{i=1 \ldots m} a_i = b_{s(i)}\} \)
(Note \( \sigma_1 \subseteq \sigma_2 \Rightarrow V_s(\sigma_1) \subseteq V_s(\sigma_2) \).)

\( \textbf{L2} \) If \( \bar{c} \in \mu_\Gamma(\chi_n) \), then \( g_{\bar{c}} \in \text{Pol} \Gamma \).

Proof. Assume, \( g_{\bar{c}} \notin \text{Pol} \Gamma \). Then there exist \( \varrho \in \Gamma \) and \( \bar{r}_1, \ldots, \bar{r}_n \in \varrho \) such that \( \bar{b} := g_{\bar{c}}(\bar{r}_1, \ldots, \bar{r}_n) \notin \varrho \). Let \( m \) be the arity of \( \varrho \). For every \( n \)-tuple \( (\bar{r}_1(i), \ldots, \bar{r}_n(i)) \) there is an \( s(i) \in \{1, \ldots, |A|^n\} \) such that \( (\bar{r}_1(i), \ldots, \bar{r}_n(i)) = (\bar{q}_1(s(i)), \ldots, \bar{q}_n(s(i))) \) for all \( i \). This implies \( \bar{q}_j \in W_s(\varrho) \) and therefore \( \chi_n \subseteq W_s(\varrho) \) and \( \mu_\Gamma(\chi_n) \subseteq W_s(\varrho) \).

On the other hand,
\[
\bar{b}(i) = g_{\bar{c}}(\bar{r}_1(i), \ldots, \bar{r}_n(i)) = g_{\bar{c}}(\bar{q}_1(s(i)), \ldots, \bar{q}_n(s(i))) = \bar{c}(s(i))
\]
then \( \bar{b} \notin \varrho \) implies \( \bar{c} \notin W_s(\varrho) \). Consequently, because of \( \mu_\Gamma(\chi_n) \subseteq W_s(\varrho) \), we have \( \bar{c} \notin \mu_\Gamma(\chi_n) \). \( \blacksquare \)

Now let \( \Gamma_0 \) be a relational clone and \( \varrho \in \text{Inv Pol} \Gamma_0 \) with arity \( m \).

If \( \varrho = \emptyset \), then \( \varrho \in \Gamma_0 \), because relational clones contain all empty relations. Otherwise let \( \varrho = \{\bar{r}_1, \ldots, \bar{r}_n\} \) with \( \bar{r}_j \in A^m \). As before, there is a function \( s : \{1, \ldots, m\} \to \{1, \ldots, |A|^n\} \) with \( (\bar{r}_1(i), \ldots, \bar{r}_n(i)) = (\bar{q}_1(s(i)), \ldots, \bar{q}_n(s(i))) \) for all \( i \leq m \). This implies \( \bar{r}_j \in V_s(\chi_n) \) and \( \varrho \subseteq V_s(\chi_n) \subseteq V_s(\mu_{\Gamma_0}(\chi_n)) \). Therefore
\[
\mu_{\Gamma_0}(\varrho) \subseteq \mu_{\Gamma_0}(V_s(\mu_{\Gamma_0}(\chi_n))) = V_s(\mu_{\Gamma_0}(\chi_n))
\]
Let \( \bar{b} \in \mu_{\Gamma_0}(\varrho) \). Then \( \bar{b} \in V_s(\mu_{\Gamma_0}(\chi_n)) \) and therefore there exists \( \bar{c} \in \mu_{\Gamma_0}(\chi_n) \) with \( \bar{b}(i) = \bar{c}(s(i)) \) for all \( i \leq m \). But by definition we have \( \bar{c}(s(i)) = g_{\bar{c}}(\bar{q}_1(s(i)), \ldots, \bar{q}_n(s(i))) = g_{\bar{c}}(\bar{r}_1(i), \ldots, \bar{r}_n(i)) \). Consequently
\[
\bar{b} = g_{\bar{c}}(\bar{r}_1, \ldots, \bar{r}_n).\]
By $L^2$, $g_\varepsilon$ is in $\text{Inv} \Gamma_0$. Now $\varrho \in \text{Pol Inv} \Gamma_0$ implies that $\varrho$ is invariant for $g_\varepsilon$. This show $\tilde{b} \in \varrho$.

But then $\varrho = \mu_{\Gamma_0}(\varrho)$, and this is equivalent with $\varrho \in \Gamma_0$. $\blacksquare$