Some other Galois connections

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A : fixed <u>finite</u> set, $|A| \geq 2$

Galois connection Inv – Pol

between $\mathcal{P}\,\mathbf{O}_A$ and $\mathcal{P}\,\mathbf{R}_A$

based on "$f \operatorname{pres} \varrho$"

$\mathsf{Pol}\,\mathsf{Inv}\,F = \mathsf{clone}(F)$

$\mathsf{Inv}\,\mathsf{Pol}\,\Gamma = \langle\Gamma\rangle_{\exists,\wedge,=,\mathsf{false}}$

From this connection we can obtain other Galois connections, also based on the relationship pres, by restricting or extending the sets $\mathbf{O}_A$ or $\mathbf{R}_A$.

Example

Let $\mathbf{sO}_A := \{f \in \mathbf{O}_A \mid f \text{ surjective }\}$

For $\Gamma \subseteq \mathbf{R}_A$ put $\mathsf{sPol}\,\Gamma := \mathbf{sO}_A \cap \mathsf{Pol}\,\Gamma$.

Then Inv – sPol is a Galois connection between $\mathcal{P}\,\mathbf{sO}_A$ and $\mathcal{P}\,\mathbf{R}_A$

$\mathsf{sPol}\,\mathsf{Inv}\,F = \mathbf{sO}_A \cap \mathsf{clone}(F)$
= surjective part of a clone

<u>better</u>: $\mathsf{sPol}\,\Gamma \mapsto \mathsf{clone}(\mathsf{sPol}\,\Gamma) = \mathsf{Pol}\,\mathsf{Inv}\,\mathsf{sPol}\,\Gamma$

A clone C is *surjectively generated*

$:\iff \quad C = \mathsf{clone}(\mathbf{sO}_A \cap C)$

($\mathbf{sO}_A \cap \mathsf{Pol}\,\mathsf{Inv}\,\mathsf{sPol}\,\Gamma = \mathsf{sPol}\,\Gamma$,

$\mathsf{Inv}(\mathsf{Pol}\,\mathsf{Inv}\,\mathsf{sPol}\,\Gamma) = \mathsf{Inv}\,\mathsf{sPol}\,\Gamma$)

$\mathsf{Inv}\,\mathsf{sPol}\,\Gamma = \langle\Gamma\rangle_{\forall,\exists,\wedge,=}$

The Quantified CSP

Let $\Gamma \subseteq \mathbf{R}_A$ be a finite set of relations. Then $\mathrm{QCSP}(\Gamma)$ denotes the following decision problem:

Input: A sentence of the form

$$(Q_1 x_1) \dots (Q_n x_n) \, (\varrho_1(x_{i_{1,1}}, \dots) \wedge \dots \wedge \varrho_m(x_{i_{m,1}}, \dots) \,)$$

with $\varrho_j \in \Gamma$, $i_{s,t} \in \{1, \dots, n\}$, $Q_j \in \{\forall, \exists\}$.

Problem: Is this sentence true?

Assume $\Gamma_2 \subseteq \langle \Gamma_1 \rangle_{\forall,\exists,\wedge,=}$.

Then for every relation $\sigma \in \Gamma_2$ there is a formula $\varphi \in \Phi(\forall, \exists, \wedge, =)$ with

$$\sigma(x_1, \ldots, x_m) \iff \varphi(\varrho_1, \ldots, \varrho_l; x_1, \ldots, x_m),$$

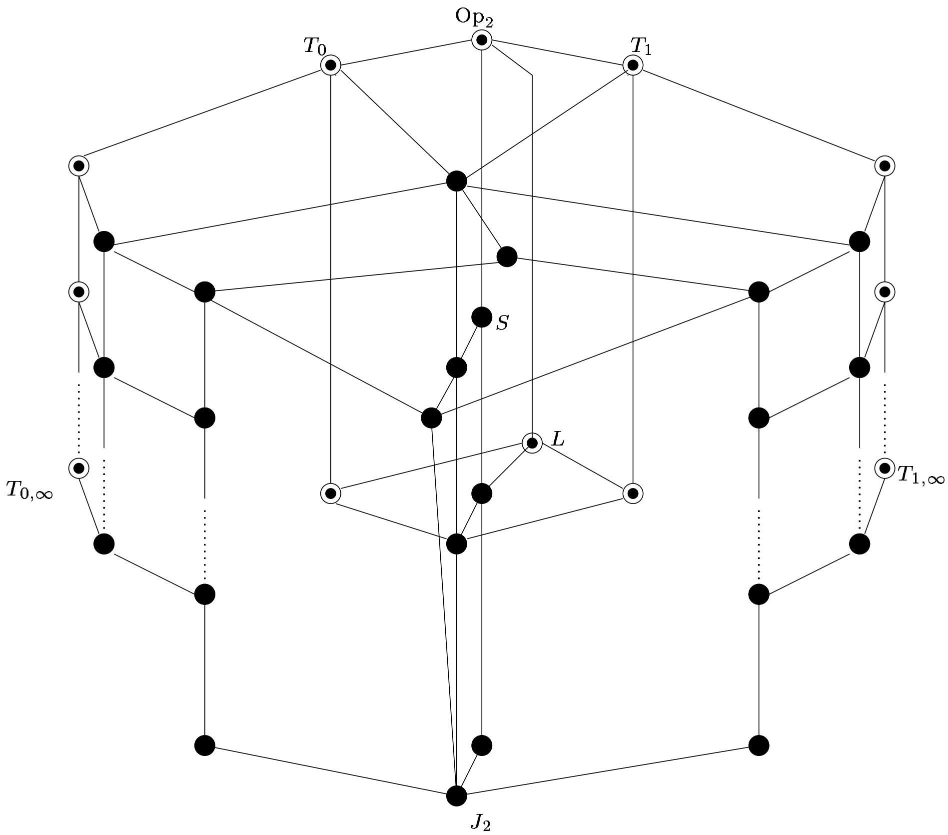
where $\varrho_1, \ldots, \varrho_l \in \Gamma_1$. If we have an input of $\mathrm{QCSP}(\Gamma_2)$, then the $\sigma \in \Gamma_2$ can be replaced by the formulas on the right side. Then the new sentence can be transformed into a normal form that corresponds to an instance of $\mathrm{QCSP}(\Gamma_1)$.

It can be shown that this is possible in polynomial time.

$$\Gamma_2 \subseteq \langle \Gamma_1 \rangle_{\forall,\exists,\wedge,=} \iff \Gamma_2 \subseteq \mathsf{Inv}\,\mathsf{sPol}\,\Gamma_1$$
$$\iff \mathsf{sPol}\,\Gamma_1 \subseteq \mathsf{sPol}\,\Gamma_2$$

Theorem. *Let* $\Gamma_1, \Gamma_2 \subseteq \mathbf{R}_A$ *be finite sets of relations. If* $\mathsf{sPol}\,\Gamma_1 \subseteq \mathsf{sPol}\,\Gamma_2$, *then* $\mathrm{QCSP}(\Gamma_2)$ *is polynomial-time reducible to* $\mathrm{QCSP}(\Gamma_1)$.

If $\mathsf{sPol}\,\Gamma_1 = \mathsf{sPol}\,\Gamma_2$, *then* $\mathrm{QCSP}(\Gamma_1)$ *and* $\mathrm{QCSP}(\Gamma_2)$ *are polynomial-time equivalent.*



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Restricting $\mathbf{O}_A$

Restricting $\mathbf{O}_A$ to	Galois connection	closed sets of functions	closure $\mathsf{Inv}\ldots\Gamma$	closed sets of relations
$\mathbf{O}_A$	Pol – Inv	clones	$\langle\Gamma\rangle_{\exists,\wedge,=,\mathsf{false}}$	relation. clones
$\mathbf{sO}_A$	sPol – Inv	surj. part of clones	$\langle\Gamma\rangle_{\forall,\exists,\wedge,=}$	universally closed relation. clones
$\mathbf{O}_A^{(n)}$	$\mathsf{Pol}^{(n)}$ – Inv	n-ary part of clones	$\mathsf{LOC}_n\langle\Gamma\rangle_{\exists,\wedge,=,\mathsf{false}}$	n-closed relation. clones
$\mathbf{O}_A^{(1)}$	End – Inv	transformation monoids	$\langle\Gamma\rangle_{\exists,\wedge,\vee,=,\mathsf{false}}$	weak Krasner algebras
$\mathbf{S}_A$	Aut – Inv	permutation groups	$\langle\Gamma\rangle_{\forall,\exists,\neg,\wedge,\vee,=,\neq}$	Krasner algebras

$$\mathsf{LOC}_n\,\Gamma := \{\varrho \in \mathbf{R}_A \mid (\forall \varrho_0 \subseteq \varrho)$$
$$(|\varrho_0| \leq n \Rightarrow (\exists \sigma \in \Gamma)(\varrho_0 \subseteq \sigma \subseteq \varrho\}$$

Extending $\mathbf{O}_A$

Extending $\mathbf{O}_A$ to	Galois connection	closed sets of functions	closure $\mathsf{Inv}\ldots\Gamma$	closed sets of relations
$\mathbf{Par}_A$	$\mathsf{pPol} - \mathsf{Inv}$	strong partial clones	$\langle\Gamma\rangle_{\wedge,=,\mathsf{false}}$	weak systems with 0 and id.
$\mathbf{pHyp}_A$	$\mathsf{pkPol} - \mathsf{Inv}$	strong clones of partial hyperf.	$\langle\Gamma\rangle_{\wedge,\mathsf{false}}$	weak systems with 0
$\mathbf{Hyp}_A$	$\mathsf{kPol} - \mathsf{Inv}$	strong clones of (total) hyperf.	$\langle\Gamma\rangle_{\exists,\wedge,\mathsf{false}}$	weak systems with proj. and 0
$\mathbf{pHyp}_A^{(1-1)}$	$\mathsf{smEnd} - \mathsf{sInv}$	strong involut. monoids	$\langle\Gamma\rangle_{\wedge,\vee,\neg}$	Boolean systems

$\mathbf{pHyp}_A := \{f \mid f : A^n \to \mathcal{P}\, A,\ n \in \mathbf{N}^+\}$
partial hyperfunctions

$\mathbf{Hyp}_A := \{f \mid f : A^n \to \mathcal{P}\, A\backslash\{\emptyset\},\ n \in \mathbf{N}^+\}$
total hyperfunctions

$\mathbf{Par}_A :=$ set of all partial functions on A

Recipes

Starting from partial hyperfunctions, there are roughly the following connections between elements of the closure on the relational side and properties of the partial hyperfunctions, used in the Galois connections.

$\wedge, \mathsf{false}$	(all partial hyperfunctions)
$=$	unique, $\lvert f(\vec{a})\rvert \leq 1$
$\neq$	injective
$\vee$	unary
$\exists$	everywhere defined, $\lvert f(\vec{a})\rvert \geq 1$
$\forall$	surjective
$\neg$	ivolutable, invertable (strong invariance)

Remarks:

$$\langle\Gamma\rangle_{\forall,\exists,\wedge,\vee,=} \quad = \quad \langle\Gamma\rangle_{\forall,\exists,\wedge,\vee,\neg,=,\neq} \text{ } (|A| \geq 2)$$

$$\langle\Gamma\rangle_{\exists,\wedge,=,\neq} \quad = \quad \langle\Gamma\rangle_{\forall,\exists,\wedge,\vee,\neg,=,\neq} \text{ if } |A| \geq 3$$

Main idea for the proof

The difficult part of the proof is:

$$\Gamma_0 \text{ relational clone } \Rightarrow \mathsf{Inv}\,\mathsf{Pol}\,\Gamma_0 \subseteq \Gamma_0$$

So let $\varrho \in \mathsf{Inv}\,\mathsf{Pol}\,\Gamma_0$ with arity m. We want to show $\varrho \in \Gamma_0$.

If $\varrho = \emptyset$, then $\varrho \in \Gamma_0$, because a relational clone contains all empty relations. Otherwise let $\varrho = \{\vec{r}_1, \ldots, \vec{r}_n\}$ with $\vec{r}_j \in A^m$.

Consider $\mu_{\Gamma_0}(\varrho) := \bigcap\{\sigma \mid \sigma \in \Gamma_0 \text{ and } \varrho \subseteq \sigma\}$

Γ_0 is $\bigcap$-closed, therefore

$$\varrho \in \Gamma_0 \iff \varrho = \mu_{\Gamma_0}(\varrho)$$

Let $\vec{b} \in \mu_{\Gamma_0}(\varrho)$ We have to show $\vec{b} \in \varrho$. Then $\varrho = \mu_{\Gamma_0}(\varrho)$ and the proof is finished.

Consider the partial assignment $(\vec{r}_1, \ldots, \vec{r}_n) \mapsto \vec{b}$, i.e.
$(\vec{r}_1(i), \ldots, \vec{r}_n(i) \mapsto \vec{b}(i)$ for all i with $1 \leq i \leq m$. The main idea in this proof, and similar also the main idea for the proofs for the other Galois connections, is the following:

Extend this partial assignment to a function g in $\mathsf{Pol}\,\Gamma_0$!

If this is possible, then $\varrho \in \mathsf{Inv}\,\mathsf{Pol}\,\Gamma_0$ implies that g preserves ϱ, and therefore $\vec{b} = g(\vec{r}_1, \ldots, \vec{r}_n) \in \varrho$.

Theorem. *Let* $C \subseteq \mathbf{O}_A$*. Then* C *is a clone of functions iff* $C = \mathsf{Pol}\,\Gamma$ *for some set* $\Gamma \subseteq \mathbf{R}_A$.

If $F \subseteq \mathbf{O}_A$*, then* $\mathsf{clone}\,F = \mathsf{Pol}\,\mathsf{Inv}\,F$.

Let $\Gamma_0 \subseteq \mathbf{R}_A$*. Then* Γ_0 *is a relational clone iff* $\Gamma_0 = \mathsf{Inv}\,F$ *for some set* $F \subseteq \mathbf{O}_A$.

If $\Gamma \subseteq \mathbf{R}_A$*, then* $\langle \Gamma \rangle = \mathsf{Inv}\,\mathsf{Pol}\,\Gamma$.

What remains to prove?

C clone $\iff$ $C = \mathsf{Pol}\,\mathsf{Inv}\,C$

Γ_0 relational clone $\iff$ $\Gamma_0 = \mathsf{Inv}\,\mathsf{Pol}\,\Gamma_0$

The "$\Leftarrow$" directions are easy.

$C \subseteq \mathsf{Pol}\,\mathsf{Inv}\,C$, $\Gamma_0 \subseteq \mathsf{Inv}\,\mathsf{Pol}\,\Gamma_0$ clear

Remains:

C clone $\Rightarrow \mathsf{Pol}\,\mathsf{Inv}\,C \subseteq C$

Γ_0 relational clone $\Rightarrow \mathsf{Inv}\,\mathsf{Pol}\,\Gamma_0 \subseteq \Gamma_0$

Let $\sigma \subseteq A^m$, $\Gamma \subseteq \mathbf{R}_A$

$\mu_\Gamma(\sigma) := \bigcap \{\varrho \mid \varrho \in \Gamma \text{ and } \sigma \subseteq \varrho\}$

- $\sigma \subseteq \mu_\Gamma(\sigma)$
- Γ $\bigcap$-closed $\Rightarrow \mu_\Gamma(\sigma) \in \Gamma$
$$\sigma \in \Gamma \iff \sigma = \mu_\Gamma(\sigma)$$
- $\sigma_1 \subseteq \sigma_2 \Rightarrow \mu_\Gamma(\sigma_1) \subseteq \mu_\Gamma(\sigma_2)$

L1 *Let* $\sigma = \{\vec{r}_1, \ldots, \vec{r}_n\}$*. Then*

$$\mu_{\mathsf{Inv}\,F}(\sigma) = \{f(\vec{r}_1, \ldots, \vec{r}_n) \mid f \in \mathsf{clone}^{(n)}F\}$$

Proof: The right side contains σ, is in $\mathsf{Inv}\,F$, and is the least invariant relation that contains $\vec{r}_1, \ldots, \vec{r}_n$. ∎

χ_n : $|A|^n$-ary relation on A

- Write down all n-tuples on a as rows of a matrix (lexicographic).
- The n columns of this matrix are the elements $\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_n$ of χ_n.
- For $i = 1 \ldots |A|^n$, $(\vec{q}_1(i), \vec{q}_2(i), \ldots, \vec{q}_n(i))$ runs through all possible n-tuples.

$$A = \{0,1\},\, n = 3,\, \chi_3 = \begin{array}{ccc} \vec{q}_1 & \vec{q}_2 & \vec{q}_3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{array}$$

If $\vec{c}$ is an $|A|^n$-tuple, then we can define an n-ary function $g_{\vec{c}}$ by
$g(\vec{q}_1(i), \ldots, \vec{q}_n(i)) := \vec{c}(i)$ for all $i = 1, \ldots, |A|^n$.

If $\vec{c} \in \mu_{\mathsf{Inv}\, C}(\chi_n)$, then by L1, $g_{\vec{c}} \in \mathsf{clone}^{(n)}\, C = C^{(n)}$.

Let $f \in \mathsf{Pol}\,\mathsf{Inv}\, C$ with arity n.

Then f preserves $\mu_{\mathsf{Inv}\, C}(\chi_n) \in \mathsf{Inv}\, C$. Consequently,
$\vec{c} := f(\vec{q}_1, \ldots, \vec{q}_n) \in \mu_{\mathsf{Inv}\, C}(\chi_n)$ and $f = g_{\vec{c}}$. Therefore $f \in C$. This shows that for a clone C always $\mathsf{Pol}\,\mathsf{Inv}\, C \subseteq C$.

It remains to proof $\mathsf{Inv}\,\mathsf{Pol}\,\Gamma_0 \subseteq \Gamma_0$ for each relational clone Γ_0.

Let $s : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, n\}$

$W_s : \mathbf{R}_A^{(m)} \to \mathbf{R}_A^{(n)}$

$W_s(\sigma) := \{(a_1, \ldots, a_n) \mid \sigma(a_{s(1)}, \ldots, a_{s(m)})\}$

$V_s : \mathbf{R}_A^{(n)} \to \mathbf{R}_A^{(m)}$

$V_s(\sigma) := \{(a_1, \ldots, a_m) \mid (\exists b_1) \ldots (\exists b_n)\sigma(b_1, \ldots, b_n) \wedge$
$\wedge \bigwedge_{i=1\ldots m} a_i = b_{s(i)}\}$

(Note $\sigma_1 \subseteq \sigma_2 \Rightarrow V_s(\sigma_1) \subseteq V_s(\sigma_2)$.)

L2 *If $\vec{c} \in \mu_\Gamma(\chi_n)$, then $g_{\vec{c}} \in \mathsf{Pol}\,\Gamma$.*

Proof. Assume, $g_{\vec{c}} \notin \mathsf{Pol}\,\Gamma$. Then there exist $\varrho \in \Gamma$ and $\vec{r}_1, \ldots, \vec{r}_n \in \varrho$ such that $\vec{b} := g_{\vec{c}}(\vec{r}_1, \ldots, \vec{r}_n) \notin \varrho$. Let m be the arity of ϱ. For every n-tuple $(\vec{r}_1(i), \ldots, \vec{r}_n(i))$ there is an $s(i) \in \{1, \ldots, |A|^n\}$ such that $(\vec{r}_1(i), \ldots, \vec{r}_n(i)) = (\vec{q}_1(s(i)), \ldots, \vec{q}_n(s(i)))$ for all i. This implies $\vec{q}_j \in W_s(\varrho)$ and therefore $\chi_n \subseteq W_s(\varrho)$ and $\mu_\Gamma(\chi_n) \subseteq W_s(\varrho)$.

On the other hand,

$$\vec{b}(i) = g_{\vec{c}}(\vec{r}_1(i), \ldots, \vec{r}_n(i)) = g_{\vec{c}}(\vec{q}_1(s(i)), \ldots, \vec{q}_n(s(i))) = \vec{c}(s(i))$$

then $\vec{b} \notin \varrho$ implies $\vec{c} \notin W_s(\varrho)$. Consequently, because of $\mu_\Gamma(\chi_n) \subseteq W_s(\varrho)$, we have $\vec{c} \notin \mu_\Gamma(\chi_n)$. ■

Now let Γ_0 be a relational clone and $\varrho \in \mathsf{Inv}\,\mathsf{Pol}\,\Gamma_0$ with arity m.

If $\varrho = \emptyset$, then $\varrho \in \Gamma_0$, because relational clones contain all empty relations. Otherwise let $\varrho = \{\vec{r}_1, \ldots, \vec{r}_n\}$ with $\vec{r}_j \in A^m$. As before, there is a function $s : \{1, \ldots, m\} \to \{1, \ldots, |A|^n\}$ with $(\vec{r}_1(i), \ldots, \vec{r}_n(i)) = (\vec{q}_1(s(i)), \ldots, \vec{q}_n(s(i)))$ for all $i \leq m$. This implies $\vec{r}_j \in V_s(\chi_n)$ and $\varrho \subseteq V_s(\chi_n) \subseteq V_s(\mu_{\Gamma_0}(\chi_n))$. Therefore

$$\mu_{\Gamma_0}(\varrho) \subseteq \mu_{\Gamma_0}(V_s(\mu_{\Gamma_0}(\chi_n))) = V_s(\mu_{\Gamma_0}(\chi_n))$$

Let $\vec{b} \in \mu_{\Gamma_0}(\varrho)$. Then $\vec{b} \in V_s(\mu_{\Gamma_0}(\chi_n))$ and therefore there exists $\vec{c} \in \mu_{\Gamma_0}(\chi_n)$ with $\vec{b}(i) = \vec{c}(s(i))$ for all $i \leq m$. But by definition we have $\vec{c}(s(i)) = g_{\vec{c}}(\vec{q}_1(s(i)), \ldots, \vec{q}_n(s(i))) = g_{\vec{c}}(\vec{r}_1(i), \ldots, \vec{r}_n(i))$. Consequently

$$\vec{b} = g_{\vec{c}}((\vec{r}_1, \ldots, \vec{r}_n).$$

By L2, $g_{\vec{c}}$ is in $\mathsf{Inv}\,\Gamma_0$. Now $\varrho \in \mathsf{Pol}\,\mathsf{Inv}\,\Gamma_0$ implies that ϱ is invariant for $g_{\vec{c}}$. This show $\vec{b} \in \varrho$.

But then $\varrho = \mu_{\Gamma_0}(\varrho)$, and this is equivalent with $\varrho \in \Gamma_0$. ∎