The Complexity of Temporal Constraint Satisfaction Problems

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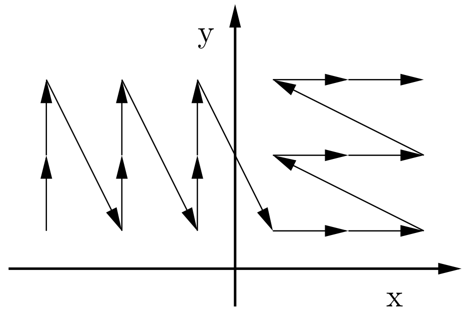
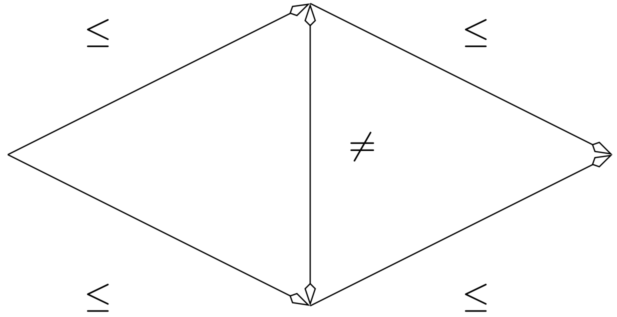


History of Temporal Reasoning

The Point Algebra (Villain+Kautz'87, van Beek'90):
Variables: denote rational numbers
Constraints: of the form $x = y$, $x < y$, $x \neq y$, $x \leq y$.

Example



Computational problem: given a set of constraints on a set of variables, are the constraints consistent?

Remarks A linear-time algorithm is known.

Ord-Horn

Definition [Nebel+Bürkert, JACM'95]: The temporal constraint language Ord-Horn consists of all relations definable as

$$(x_1 = y_1 \wedge \cdots \wedge x_k = y_k) \rightarrow x \leq y \quad \text{or}$$
$$(x_1 = y_1 \wedge \cdots \wedge x_k = y_k) \rightarrow x \neq y$$

Remarks Ord-Horn contains the Point Algebra
Was studied in the context of Allen's Interval Algebra

Theorem Consistency of Ord-Horn constraints can be decided in cubic time (by resolution)

Constrained Ordering Problems

1. Cyclic-ordering

Given	Variables V, set of triples $(x, y, z) \in V^3$
Question	Is there an assignment for V s.t. for each triple either $x < y < z$ or $y < z < x$ or $z < x < y$?

2. Betweenness

Given	Variables V, set of triples $(x, y, z) \in V^3$
Question	Is there an assignment for V s.t. for each triple either $x < y < z$ or $z < y < x$?

3. Max-constrained Ordering

Given	Variables V, set of triples $(x, y, z) \in V^3$
Question	Is there an assignment for V s.t. for each triple either $x < y$ or $x < z$?

The General Problem

Let $\Gamma = (\mathbb{Q}; R_1, R_2, \ldots)$ be a relational structure with a first-order definition in $(\mathbb{Q}; <)$.
$\{R_1, R_2, \ldots\}$: (temporal constraint) language L

CSP(Γ):

Given	An L-sentence Φ of the form $\exists \overline{x}.\ \psi_1 \wedge \cdots \wedge \psi_l$ where ψ_i are atomic L-formulas
Question	Is Φ true in Γ?

Equivalently:

Given	A finite structure S with language L
Question	Is there a homomorphism from S to Γ?

Remarks on the General Problem

Γ: temporal constraint language

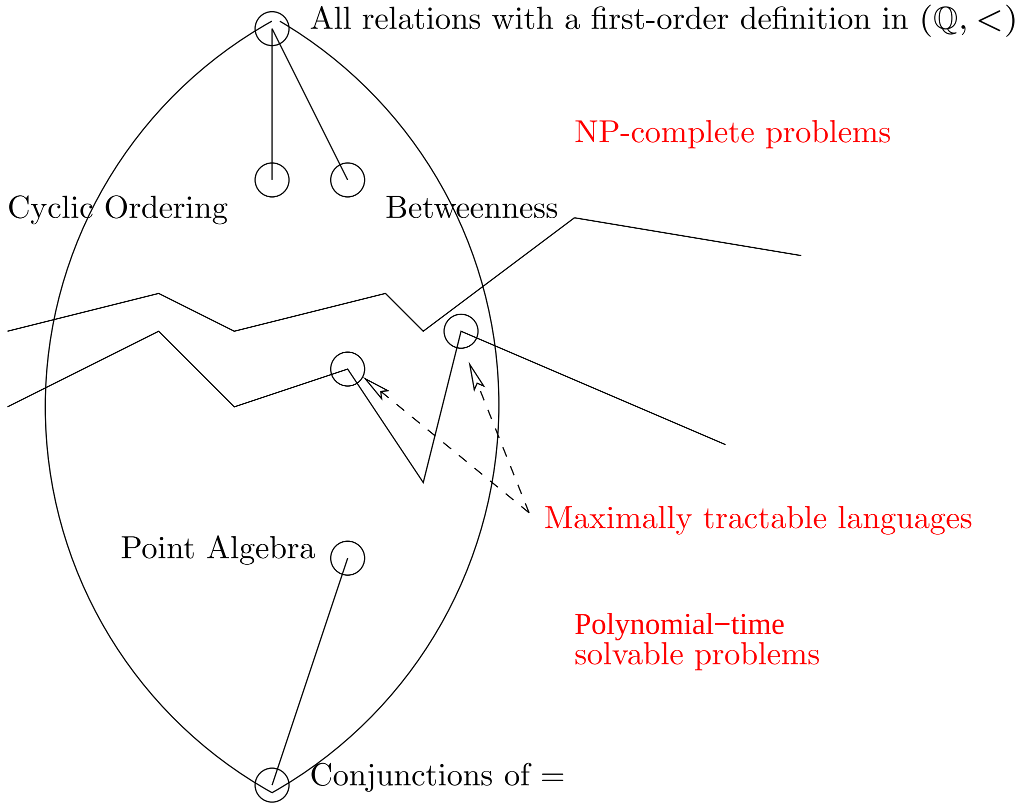
Remark 1 All problems CSP(Γ) are in NP

Remark 2 There are uncountably many constraint languages Γ

Definition A formula is primitive positive if it is of the form

$\exists \overline{x}. \psi_1 \wedge \cdots \wedge \psi_l$ where $\psi_1, \ldots, \psi_l$ are atomic.

Remark 3 Expansions of Γ by primitive positive definable relations preserve computational complexity of CSP(Γ)


All relations with a first-order definition in $(\mathbb{Q}, <)$
NP-complete problems
Cyclic Ordering
Betweenness
Maximally tractable languages
Point Algebra
Polynomial−time
solvable problems
Conjunctions of $=$

Outline

1. (Universal-) algebraic approach
2. Definition of ll-closed constraints
3. ll-closed constraints are tractable
4. The product Ramsey theorem
5. ll-closed constraints are maximally tractable
6. ll-closed constraints are not in Datalog

Polymorphisms

Definition A polymorphism f of Γ is a homomorphism $\Gamma^k \to \Gamma$ (f preserves all relations in Γ)

Examples $(x, y) \mapsto max(x, y)$ and $(x, y, z) \mapsto median(x, y, z)$ are polymorphisms of $(\mathbb{Q}; <)$

Observation Set of polymorphisms Pol(Γ) is local clone

1. Every projection is a polymorphism
2. Polymorphisms are closed under composition
3. Pol(Γ) is locally closed: If for all finite A there is $g \in$ Pol(Γ) s.t. $f(x) = g(x)$ for all $x \in A$, then $f \in$ Pol(Γ)

Working with Polymorphisms

f, g: operations from $\mathbb{Q}^k \to \mathbb{Q}$.

Definition We say that f generates g if g is contained in the smallest locally closed clone that contains f and the automorphisms of $(\mathbb{Q}, <)$.

Observation For temporal languages: polymorphisms that induce the same weak linear order on $\mathbb{Q}^k$ generate each other

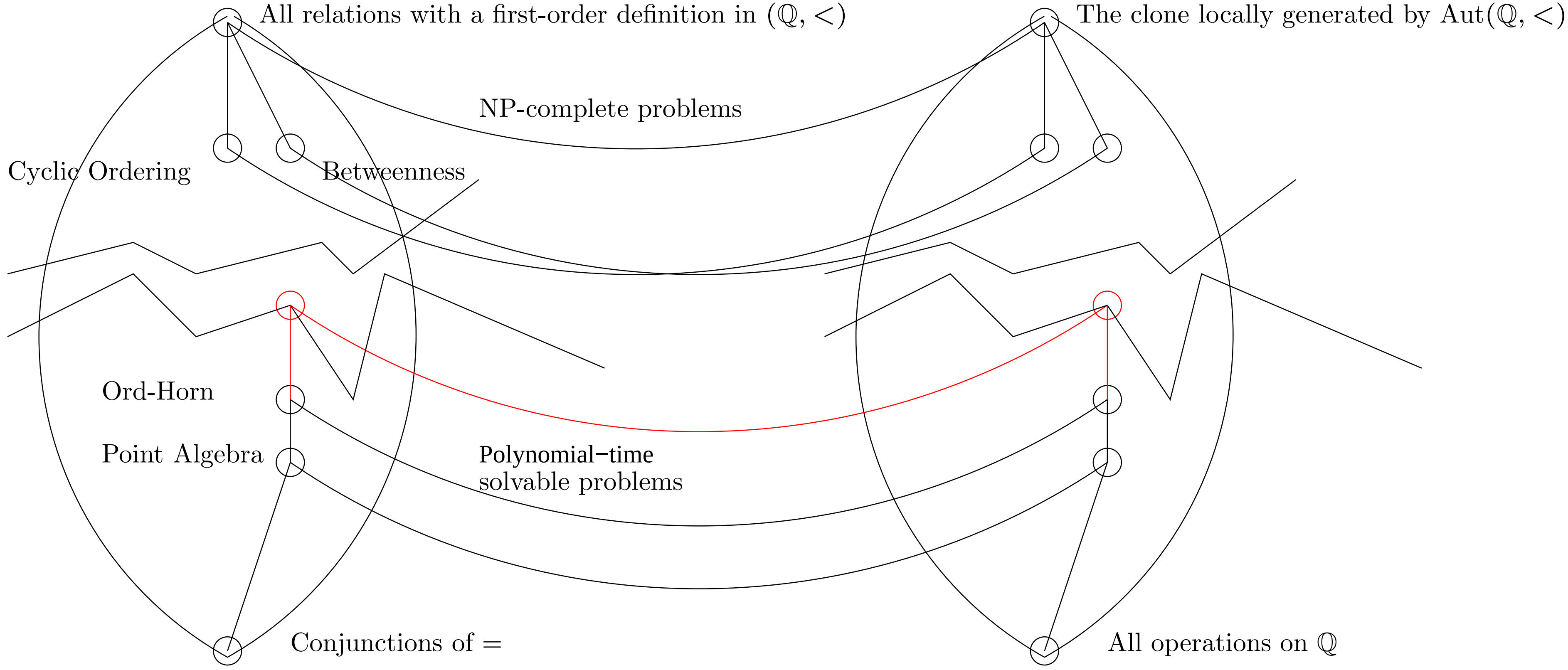
The (Universal-) Algebraic Approach

Let Γ be first-order definable in $(\mathbb{Q}; <)$

Theorem [B.+Nesetril'03]: A relation has a primitive positive definition in Γ if and only if it is preserved by all polymorphisms of Γ.

Corollary The polymorphism clone of Γ captures the computational complexity of CSP(Γ).

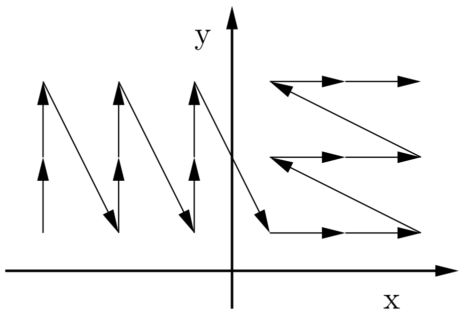
Informally More polymorphisms $\rightarrow$ less pp-definable relations
$\rightarrow$ CSP is simpler


All relations with a first-order definition in $(\mathbb{Q}, <)$
The clone locally generated by $\mathrm{Aut}(\mathbb{Q}, <)$
NP-complete problems
Cyclic Ordering
Betweenness
Ord-Horn
Point Algebra
Polynomial−time
solvable problems
Conjunctions of $=$
All operations on $\mathbb{Q}$

The II Operation

Definition — Let II be a binary operation on $\mathbb{Q}$ such that
$\text{II}(a,b) < \text{II}(a',b')$ iff
$a < 0$ and $a < a'$
$a < 0$ and $a = a'$ and $b < b'$
$a \geq 0$ and $b < b'$, or
$a \geq 0$ and $b = b'$ and $a < a'$

Illustration



Remarks — Similarly, there is a *dual II* operation

Observation — Operations generated by II are injective.

What are the Relations Preserved by ll?

Observation A k-ary temporal relation can be represented as a set of weak linear orders on $[1, \ldots, k]$ elements.

Examples Betweenness can be represented as

$$\{[1,2,3],[3,2,1]\}$$

The relation $x{=}y \vee y{=}z$ can be represented as

$$\{[0,0,1],[1,1,0],[0,1,1],[1,0,0],[0,0,0]\}$$

ll-closure Consider a 9-ary relation R

$$[-2,-1,-1,+1,+2,+2,+3,+4,+4] \in R$$
$$[+3,+3,+4,+4,+1,+1,-3,-2,-1] \in R$$

then $$[-3,-2,-1,+5,+4,+4,+1,+2,+3] \in R$$

Examples of Π-closed Constraints

1. The relations from the point algebra: $\leq$, $\neq$, $<$
2. Ord-Horn relations, e.g., the 4-ary relation defined by $(x = y \wedge u = v) \rightarrow x \leq v$, or by $(x = y \wedge u = v) \rightarrow x \neq v$
3. The relation defined by $x > y \vee x > z$.
4. The relation represented by $\{[0, 0, 1, 1], [0, 1, 2, 3]\}$.

An Algorithm for Il-closed Constraints

Theorem Il-closed constraints can be solved in quadratic time

Lemma There is an algorithm Spec that decides in linear time whether a given set of Il-closed constraints has an injective solution, no solution, and otherwise returns (x, y) such that $x = y$ in all solutions.

Algorithm

Solve(S):
If Spec(S) $=$ false return false
If Spec(S) $=$ true return true
If Spec(S) $= (x, y)$ then
  Let S' be S where x,y are contracted
  Solve(S')

Remark Also obtain new algorithm for Ord-Horn

Maximal Tractability

Suppose Γ is a language that strictly contains II.
All polymorphisms f of Γ are generated by II.

Obs. 1 f is injective.
Obs. 2 f preserves $\leq$.

It suffices to prove:

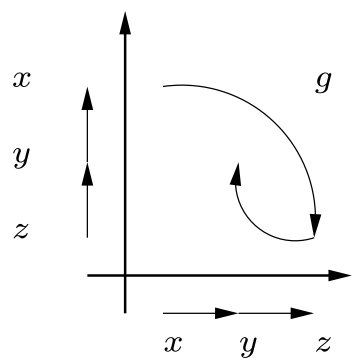
Claim Either f preserves Betweenness, or f generates II.

Violating Betweenness

Lemma If f violates Betweenness, then f generates a binary operation g that violates Betweenness.

Proof Suppose $t_1, \ldots, t_k \in$ Betw, but $f(t_1, \ldots, t_k) \notin$ Betw. Suppose wlog that $t_1(1) < t_1(2) < t_1(3)$ and $t_2(3) < t_2(2) < t_2(1)$. Then $g(x, y) := f(x, y, \alpha_3(z_1), \ldots, \alpha_k(z_k))$ violates R for appropriate automorphisms $\alpha_3, \ldots, \alpha_k$ and $z_1, \ldots, z_k \in \{x, y\}$.

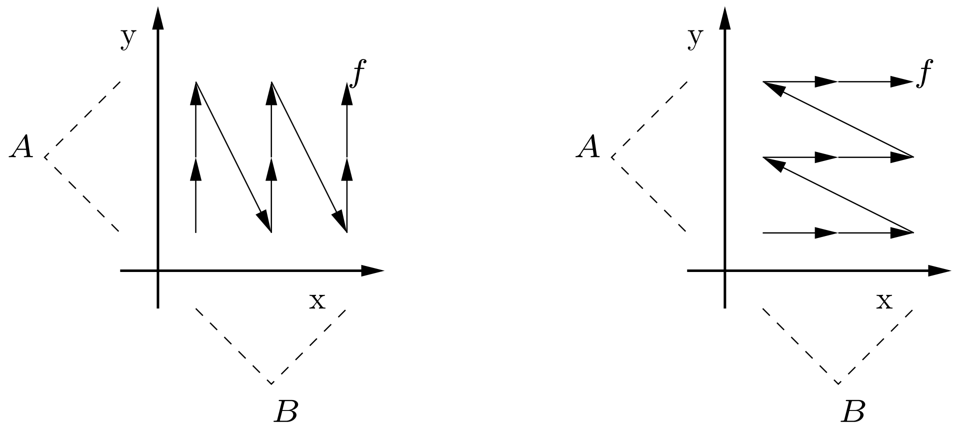
Illustration The following operation g violates Betweenness:



The Product Ramsey Theorem

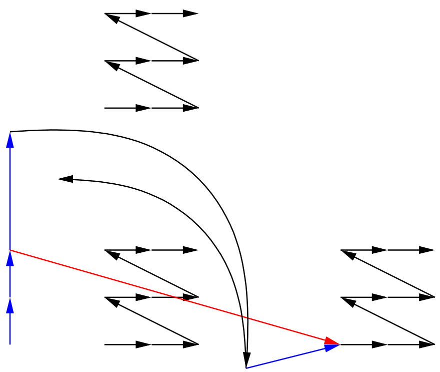
Theorem (PRT; from Trotter'01) Let k, r, and d be positive integers, and $S_1, \ldots, S_d$ be infinite sets. If $S_1 \times \cdots \times S_d$ is linearly ordered, and the $[k]^d$-subgrids of $S_1 \times \cdots \times S_d$ are colored with r colors such that $[k]^d$-subgrids inducing isomorphic linear orders get the same color, then there exist infinite sets $S_i' \subset S_i$ such that all $[k]^d$-subgrids of $S_1' \times \cdots \times S_d'$ have the same color.

Application For every binary injective operation f on $\mathbb{Q}$ preserving $\leq$ there are two infinite subsets $A, B \subset \mathbb{Q}$ such that f defines a lexicographic ordering on $A \times B$.



Proof Sketch, ctnd

Main step Let f be binary injective such that f preserves $\leq$ but not Betweenness. By appropriately applying the PRT three times we obtain the following picture.



Final step If a relation R is preserved by an operation with the 'pattern above', it is preserved by ll.

Datalog

Example Datalog program for CSP$(\mathbb{Q}, <)$:

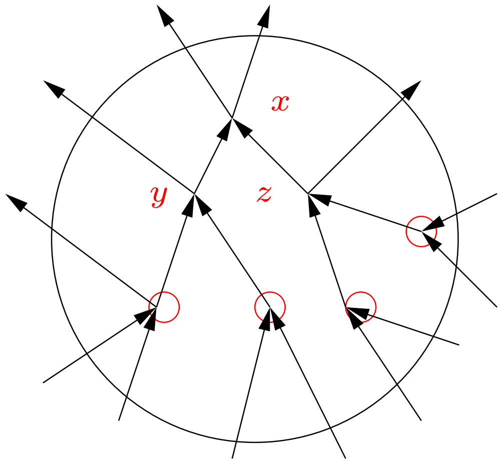
$$\begin{array}{rcl} \text{tc}(x,y) & \leftarrow & x<y \\ \text{tc}(x,y) & \leftarrow & \text{tc}(x,u),\ \text{tc}(u,y) \\ \text{false} & \leftarrow & \text{tc}(x,x) \end{array}$$

Question Is there a Datalog program
that solves ll-closed constraints?

Theorem (B.+Dalmau'06) CSP$(\Gamma) \notin$ Datalog
iff for every k there is an inconsistent instance S
s.t. Duplicator wins the existential k-pebble game on S, Γ

The Inconsistent Instance

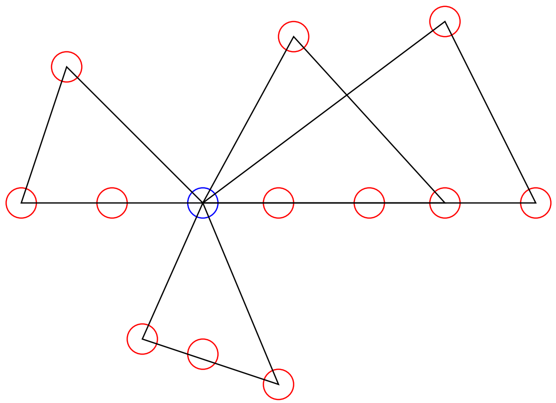
1 Take a 4-regular graph of girth $\geq 2k$ (exist...)
2 Orient the edges according to an Euler tour
3 Transform graph to an instance of CSP$(\mathbb{Q}, x > y \vee x > z)$



Def A binary subtree of the graph is called dominated if all leafes are pebbled

The Winning Strategy for Duplicator

Duplicator always tries to pebble roots of dominated trees by a value larger than the min of all leafes



Obs 1 Duplicator can always do that.

Obs 2 In that way, she can always preserve all relations

Outlook

Full classifiaction?

QCSP?

For universal algebraists: Classification of the locally closed clones that contain the automorphisms of $(\mathbb{Q}; <)$?

For model theorists: Classification of the reducts of $(\mathbb{Q}; <)$ up to primitive positive inter-definability?

For computer scientists: Can we extend the algorithm for ll-closed constraints to constraints like $3x + y < 7$?