The Complexity of Temporal Constraint Satisfaction Problems

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History of Temporal Reasoning

The Point Algebra (Villain+Kautz'87, van Beek’90):
Variables: denote rational numbers
Constraints: of the form $x = y$, $x < y$, $x \neq y$, $x \leq y$.

Example

\[
\begin{array}{ccc}
\leq & \leq & \leq \\
\leq & \neq & \leq \\
\leq & \leq & \leq \\
\end{array}
\]

Computational problem: given a set of constraints on a set of variables, are the constraints consistent?

Remarks A linear-time algorithm is known.
Ord-Horn

Definition [Nebel+Bürkert, JACM’95]: The temporal constraint language Ord-Horn consists of all relations definable as

\[(x_1 = y_1 \land \cdots \land x_k = y_k) \rightarrow x \leq y \] or \n\[(x_1 = y_1 \land \cdots \land x_k = y_k) \rightarrow x \neq y \]

Remarks Ord-Horn contains the Point Algebra
Was studied in the context of Allen’s Interval Algebra

Theorem Consistency of Ord-Horn constraints can be decided in cubic time (by resolution)
Constrained Ordering Problems

1. **Cyclic-ordering**
   
   **Given** Variables $V$, set of triples $(x, y, z) \in V^3$
   
   **Question** Is there an assignment for $V$ s.t. for each triple either $x < y < z$ or $y < z < x$ or $z < x < y$?

2. **Betweenness**
   
   **Given** Variables $V$, set of triples $(x, y, z) \in V^3$
   
   **Question** Is there an assignment for $V$ s.t. for each triple either $x < y < z$ or $z < y < x$?

3. **Max-constrained Ordering**
   
   **Given** Variables $V$, set of triples $(x, y, z) \in V^3$
   
   **Question** Is there an assignment for $V$ s.t. for each triple either $x < y$ or $x < z$?
The General Problem

Let $\Gamma = (\mathbb{Q}; R_1, R_2, \ldots)$ be a relational structure with a first-order definition in $(\mathbb{Q}; <)$.
\{R_1, R_2, \ldots\}: (temporal constraint) language $L$

CSP($\Gamma$):

Given An $L$-sentence $\Phi$ of the form $\exists \overline{x}. \psi_1 \land \cdots \land \psi_l$
where $\psi_i$ are atomic $L$-formulas

Question Is $\Phi$ true in $\Gamma$?

Equivalently:

Given A finite structure $S$ with language $L$
Question Is there a homomorphism from $S$ to $\Gamma$?
Remarks on the General Problem

Γ: temporal constraint language

Remark 1  All problems CSP(Γ) are in NP

Remark 2  There are uncountably many constraint languages Γ

Definition  A formula is primitive positive if it is of the form
\[ \exists \vec{x}. \psi_1 \land \cdots \land \psi_l \] where \( \psi_1, \ldots, \psi_l \) are atomic.

Remark 3  Expansions of Γ by primitive positive definable relations
preserve computational complexity of CSP(Γ)
All relations with a first-order definition in \((\mathbb{Q}, <)\)

- Polynomial-time solvable problems
- Conjunctions of =
- NP-complete problems
- Maximally tractable languages
- Cyclic Ordering
- Betweenness
- Point Algebra
Outline

1. (Universal-) algebraic approach
2. Definition of Il-closed constraints
3. Il-closed constraints are tractable
4. The product Ramsey theorem
5. Il-closed constraints are maximally tractable
6. Il-closed constraints are not in Datalog
Polymorphisms

**Definition**
A polymorphism $f$ of $\Gamma$ is a homomorphism $\Gamma^k \to \Gamma$ ($f$ preserves all relations in $\Gamma$)

**Examples**
$(x, y) \mapsto max(x, y)$ and $(x, y, z) \mapsto median(x, y, z)$ are polymorphisms of $(\mathbb{Q}; <)$

**Observation**
Set of polymorphisms $\text{Pol}(\Gamma)$ is local clone

1. Every projection is a polymorphism
2. Polymorphisms are closed under composition
3. $\text{Pol}(\Gamma)$ is locally closed: If for all finite $A$ there is $g \in \text{Pol}(\Gamma)$ s.t. $f(x) = g(x)$ for all $x \in A$, then $f \in \text{Pol}(\Gamma)$
Working with Polymorphisms

\( f, g \): operations from \( \mathbb{Q}^k \rightarrow \mathbb{Q} \).

**Definition** We say that \( f \) generates \( g \) if \( g \) is contained in the smallest locally closed clone that contains \( f \) and the automorphisms of \( (\mathbb{Q}, <) \).

**Observation** For temporal languages: polymorphisms that induce the same weak linear order on \( \mathbb{Q}^k \) generate each other.
**The (Universal-) Algebraic Approach**

Let $\Gamma$ be first-order definable in $(\mathbb{Q}; <)$

**Theorem** [B.+-Nesetril’03]: A relation has a primitive positive definition in $\Gamma$ if and only if it is preserved by all polymorphisms of $\Gamma$.

**Corollary** The polymorphism clone of $\Gamma$ captures the computational complexity of CSP($\Gamma$).

**Informally** More polymorphisms $\rightarrow$ less pp-definable relations
$\rightarrow$ CSP is simpler
Polynomial-time solvable problems

Point Algebra

Conjunctions of $=$

Ord-Horn

Cyclic Ordering

Betweenness

NP-complete problems

The clone locally generated by $\text{Aut}(\mathbb{Q}, <)$

All operations on $\mathbb{Q}$

All relations with a first-order definition in $(\mathbb{Q}, <)$

Polynomial-time solvable problems
The II Operation

Definition  Let II be a binary operation on $\mathbb{Q}$ such that

$\text{II}(a, b) < \text{II}(a', b')$ iff

- $a < 0$ and $a < a'$
- $a < 0$ and $a = a'$ and $b < b'$
- $a \geq 0$ and $b < b'$, or
- $a \geq 0$ and $b = b'$ and $a < a'$

Illustration

![Diagram showing the operation II]

Remarks  Similarly, there is a dual II operation

Observation  Operations generated by II are injective.
What are the Relations Preserved by ll?

Observation A $k$-ary temporal relation can be represented as a set of weak linear orders on $[1, \ldots, k]$ elements.

Examples Betweenness can be represented as
\[
\{[1, 2, 3], [3, 2, 1]\}
\]
The relation $x=y \lor y=z$ can be represented as
\[
\{[0, 0, 1], [1, 1, 0], [0, 1, 1], [1, 0, 0], [0, 0, 0]\}
\]

ll-closure Consider a 9-ary relation $R$
\[
[-2, -1, -1, +1, +2, +2, +3, +4, +4] \in R
\]
\[
[+3, +3, +4, +4, +1, +1, -3, -2, -1] \in R
\]
then \[
[-3, -2, -1, +5, +4, +4, +1, +2, +3] \in R
\]
Examples of II-closed Constraints

1. The relations from the point algebra: \( \leq, \neq, < \)

2. Ord-Horn relations, e.g., the 4-ary relation defined by \( (x = y \land u = v) \rightarrow x \leq v \),
   or by \( (x = y \land u = v) \rightarrow x \neq v \)

3. The relation defined by \( x > y \lor x > z \).

4. The relation represented by \{[0, 0, 1, 1], [0, 1, 2, 3]\}.
An Algorithm for ll-closed Constraints

**Theorem**  ll-closed constraints can be solved in quadratic time

**Lemma**  There is an algorithm Spec that decides in linear time whether a given set of ll-closed constraints has an injective solution, no solution, and otherwise returns \((x, y)\) such that \(x = y\) in all solutions.

**Algorithm**

**Solve***(S)*:

If Spec(*S*) = false return false
If Spec(*S*) = true return true
If Spec(*S*) = \((x, y)\) then
   Let \({S}'\) be *S* where \(x, y\) are contracted
   Solve(*S'*).

**Remark**  Also obtain new algorithm for Ord-Horn
Maximal Tractability

Suppose $\Gamma$ is a language that strictly contains ll. All polymorphisms $f$ of $\Gamma$ are generated by ll.

Obs. 1 $f$ is injective.
Obs. 2 $f$ preserves $\leq$.

It suffices to prove:

Claim Either $f$ preserves Betweenness, or $f$ generates ll.
**Violating Betweenness**

**Lemma** If $f$ violates Betweenness, then $f$ generates a binary operation $g$ that violates Betweenness.

**Proof** Suppose $t_1, \ldots, t_k \in \text{Betw}$, but $f(t_1, \ldots, t_k) \notin \text{Betw}$. Suppose wlog that $t_1(1) < t_1(2) < t_1(3)$ and $t_2(3) < t_2(2) < t_2(1)$. Then $g(x, y) := f(x, y, \alpha_3(z_1), \ldots, \alpha_k(z_k))$ violates $R$ for appropriate automorphisms $\alpha_3, \ldots, \alpha_k$ and $z_1, \ldots, z_k \in \{x, y\}$.

**Illustration** The following operation $g$ violates Betweenness:
The Product Ramsey Theorem

Theorem  (PRT; from Trotter'01) Let $k$, $r$, and $d$ be positive integers, and $S_1, \ldots, S_d$ be infinite sets. If $S_1 \times \cdots \times S_d$ is linearly ordered, and the $[k]^d$-subgrids of $S_1 \times \cdots \times S_d$ are colored with $r$ colors such that $[k]^d$-subgrids inducing isomorphic linear orders get the same color, then there exist infinite sets $S'_i \subset S_i$ such that all $[k]^d$-subgrids of $S'_1 \times \cdots \times S'_d$ have the same color.

Application  For every binary injective operation $f$ on $\mathbb{Q}$ preserving $\leq$ there are two infinite subsets $A, B \subset \mathbb{Q}$ such that $f$ defines a lexicographic ordering on $A \times B$. 

![Diagram](image-url)
**Proof Sketch, ctnd**

Main step  Let $f$ be binary injective such that $f$ preserves $\leq$ but not Betweenness. By appropriately applying the PRT three times we obtain the following picture.

![Diagram](image)

Final step  If a relation $R$ is preserved by an operation with the ‘pattern above’, it is preserved by ll.
Datalog

Example  Datalog program for CSP(\(\mathbb{Q}, <\)):

\[
\begin{align*}
tc(x, y) &\leftarrow x < y \\
tc(x, y) &\leftarrow tc(x, u), tc(u, y) \\
false &\leftarrow tc(x, x)
\end{align*}
\]

Question  Is there a Datalog program that solves \(\mathcal{I}\)-closed constraints?

Theorem  (B. Dalmau’06) CSP(\(\Gamma\)) \(\not\in\) Datalog iff for every \(k\) there is an inconsistent instance \(S\) s.t. Duplicator wins the existential \(k\)-pebble game on \(S, \Gamma\)
The Inconsistent Instance

1. Take a 4-regular graph of girth $\geq 2k$ (exist...)
2. Orient the edges according to an Euler tour
3. Transform graph to an instance of CSP($\mathbb{Q}, x > y \lor x > z$)

Def: A binary subtree of the graph is called dominated if all leafes are pebbled
The Winning Strategy for Duplicator

Duplicator always tries to pebble roots of dominated trees by a value larger than the min of all leafes

Obs 1  Duplicator can always do that.

Obs 2  In that way, she can always preserve all relations
Outlook

Full classification?

QCSP?

For universal algebraists: Classification of the locally closed clones that contain the automorphisms of \((\mathbb{Q}; <)\)?

For model theorists: Classification of the reducts of \((\mathbb{Q}; <)\) up to primitive positive inter-definability?

For computer scientists: Can we extend the algorithm for \(ll\)-closed constraints to constraints like \(3x + y < 7\)?