On the Power of k-Consistency

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Constraint Satisfaction Problems

Fix a relational vocabulary $\sigma = (R_1, \ldots, R_s)$.

Notation:

- $\mathbf{A} \rightarrow \mathbf{B}$: there is a homomorphism from $\mathbf{A}$ to $\mathbf{B}$.
- $\mathbf{A} \not\rightarrow \mathbf{B}$: there isn't.

More notation:

- $\mathrm{CSP}(\mathcal{C}, \mathcal{D}) = \{(\mathbf{A}, \mathbf{B}) \in \mathcal{C} \times \mathcal{D} : \mathbf{A} \rightarrow \mathbf{B}\}$.
- $\mathrm{CSP}(\mathcal{C}, \mathbf{B}) = \{\mathbf{A} \in \mathcal{C} : \mathbf{A} \rightarrow \mathbf{B}\}$.
- $\mathrm{CSP}(\mathbf{A}, \mathcal{D}) = \{\mathbf{B} \in \mathcal{D} : \mathbf{A} \rightarrow \mathbf{B}\}$.

Very often $\mathcal{C} = \mathcal{D} =$ All.

k-Consistency Test

Given $k \geq 1$, $\mathbf{A}$ and $\mathbf{B}$:

1. Let H be the set of all partial homomorphisms $f : \mathbf{A} \longrightarrow \mathbf{B}$ with $|\mathrm{Dom}(f)| \leq k$.
2. Remove from H every $f \in H$ with $|\mathrm{Dom}(f)| < k$ such that for some $a \in A$, there is no $g \in H$ with $f \subseteq g$ and $a \in \mathrm{Dom}(g)$.
3. Repeat (2) until H stabilizes.

- If $H \neq \emptyset$, we say the test succeeds.
- If $H = \emptyset$, we say the test fails.

Complexity of the test

Comment: This version of the k-consistency test is known as the existential k-pebble game (connection made explicit by Kolaitis and Vardi).

Note: Algorithm runs in time $O((|A||B|)^k)$. For fixed k, it is polynomial.

Theorem [Kolaitis-Panttaja]:
Given k, $\mathbf{A}$ and B, deciding if the consistency test succeeds is **EXP**-complete.

Fact and Question

Fact: If the k-consistency test fails, then $\mathbf{A} \not\to \mathbf{B}$.

Question: For fixed k,

When is the converse true?

That is,

When is the k-consistency test a sound and complete algorithm to test homomorphisms?

Problems

Width-k Problem:

Characterize all $\mathbf{B}$'s for which
k-consistency solves $\mathrm{CSP}(-, \mathbf{B})$.

k-Width Problem:

Characterize all $\mathbf{A}$'s for which
k-consistency solves $\mathrm{CSP}(\mathbf{A}, -)$.

Suffi cient and Necessary Conditions

Weakest condition sufficient condition known to date:

Theorem [Dalmau-Kolaitis-Vardi]
If $\mathrm{tw}(\mathrm{core}(\mathbf{A})) \leq k$, then k-consistency solves $\mathrm{CSP}(\mathbf{A}, -)$.

We show it is in fact the weakest possible:

Theorem [Here and now]
If $\mathrm{tw}(\mathrm{core}(\mathbf{A})) > k$, then k-consistency does not solve $\mathrm{CSP}(\mathbf{A}, -)$.

Relationship with Grohe's result

Theorem [Grohe]
If **FPT** $\neq$ **W[1]** and $\mathrm{CSP}(\mathcal{C}, -)$ is in **P**, then $\mathrm{tw}(\mathrm{core}(\mathcal{C}))$ is bounded.

In particular,

Corollary
If **FPT** $\neq$ **W[1]** and $\mathrm{CSP}(\mathcal{C}, -)$ is solved by k-consistency for some k, then $\mathrm{tw}(\mathrm{core}(\mathcal{C}))$ is bounded.

We want to:

1. Get the result for a singleton $\mathcal{C} = \{\mathbf{A}\}$.
2. Obtain the tightest possible bound (namely k).
3. Remove the complexity assumption.

Proof (for graphs)

Proof strategy

Since $\text{CSP}(\mathbf{A}) = \text{CSP}(\text{core}(\mathbf{A}))$, we may as well assume that $\mathbf{A}$ is a core. Suppose $\text{tw}(\mathbf{A}) > k$.

We need: a graph $\mathbf{B}$ such that

- $\mathbf{A} \not\to \mathbf{B}$,
- k-consistency test succeeds on $\mathbf{A}$ and $\mathbf{B}$.

Construction of $\mathbf{B}$

Based on a basic fact about graphs: the sum of the degrees is even.

Fix $a_0 \in A$ arbitrarily. The universe of $\mathbf{B}$ consists of pairs $(a, \bar{b})$ where:

- $a \in A$
- $\bar{b} \in \{0, 1\}^{d_a}$ where d_a is the degree of a.

For every a, we fix an arbitrary ordering $e_1^a, \ldots, e_{d_a}^a$ of the edges of a.

- $\bar{b}$ has an even number of ones if $a \neq a_0$
- $\bar{b}$ has an odd number of ones if $a = a_0$

Edge-set of $\mathbf{B}$

The edge-set of $\mathbf{B}$ contains all pairs $((a, \bar{b}), (a', \overline{b'}))$ such that:

- (a, a') is an edge of $\mathbf{A}$
- $\bar{b}$ and $\overline{b'}$ are *compatible* wrt the edge (a, a'). That is

$$\bar{b}[i] = \overline{b'}[i']$$

where

- $i =$ position of (a, a') in $\bar{b}$. Formally $e^a_i = (a, a')$
- $i' =$ position of (a, a') in $\overline{b'}$. Formally $e^{a'}_{i'} = (a, a')$

Part 1: $\mathbf{A} \not\to \mathbf{B}$

Def: $h : A \to B$ is node-preserving if $h(a) = (a, \bar{b})$ for some $\bar{b}$.

Fact: If there is a homomorphism h from $\mathbf{A}$ to $\mathbf{B}$, then there is one that is node-preserving.

- Note $\pi_1 : \mathbf{B} \to \mathbf{A}$.
- Hence $\pi_1 \circ h : \mathbf{A} \to \mathbf{A}$.
- Since $\mathbf{A}$ is a core, $\pi_1 \circ h$ is an automorphism of $\mathbf{A}$. Hence

$$h \circ (\pi_1 \circ h)^{-1} : \mathbf{A} \to \mathbf{B}$$

is node-preserving.

Fact: Node-preserving homomorphisms violate the even-sum-of-degrees principle.

Part 2: k-consistency succeeds on $\mathbf{A}$ and $\mathbf{B}$

Let p be a walk in $\mathbf{A}$ starting at a_0.
Let S be a set of nodes of $\mathbf{A}$ that does not contain the last node of p

Def: $p_S : S \to B$ is the partial mapping defined by

$$p_S(a) = (a, \bar{b}) \text{ where } \bar{b} \text{ is}$$

$$\bar{b}[i] = \begin{cases} 0 & \text{if } e_i^a \text{ is crossed an even number of times by } p \\ 1 & \text{if } e_i^a \text{ is crossed an odd number of times by } p \end{cases}$$

Observation: p_S is a partial homomorphism from $\mathbf{A}$ to $\mathbf{B}$.

A k-consistent set of partial homomorphisms

Let $\mathcal{H}$ consists of all p_S for *valid* choices of p and S.

What is a valid choice?

Theorem: [Robertson-Thomas]
If $\text{tw}(\mathbf{A}) \geq k$ then there exists a collection of sets $B_1, \ldots, B_r$ such that for all $1 \leq i, j \leq r$:

- B_i is connected
- $B_i \cap B_j \neq \emptyset$ or there exists an edge that intersects both B_i and B_j
- for every $S \subseteq A$ with $|S| \leq k$ there exists some B_l not touched by S.

$B_1, \ldots, B_r$ is a *bramble* of order $k + 1$

Valid choices

Def: p and S is a valid choice if:

- $|S| \leq k$
- the initial node of p is a_0
- the final node of p belongs to a B_i not covered by S

The last condition guarantees that p_S is a partial homomorphism.

It remains to prove that

If $f \in \mathcal{H}$ with $|\mathrm{Dom}(f)| < k$ and $a \in A$ then
there is $g \in \mathcal{H}$ with $f \subseteq g$ and $a \in \mathrm{Dom}(g)$.

Consistency

Let p_S be a mapping in $\mathcal{H}$ and let $a \notin S$.

- The last node u of p belongs to a B_i not covered by S,
- there exists some B_j not covered by $S \cup \{a\}$. Let v be an arbitrary element of B_j.

- Let q be a path from u to v crossing only B_i and B_j. We have
  - pq and $S \cup \{a\}$ is a valid choice,
  - $pq_{S\cup\{a\}}$ extends p_S.

Q.E.D.

Consequences

Consequences

Corollary [In fact, this is trivial: take cliques]
There exist $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots$ such that $\mathrm{CSP}(\mathbf{A}_k, -)$ is solved by k-consistency but not by $(k-1)$-consistency.

Corollary [Dalmau-Kolaitis-Vardi]:
The complexity of the k-width meta-problem is **NP**-complete.

Corollary: If φ is a conjunctive query that is equivalent to an $\exists L^{k,+}_{\infty\omega}$-sentence, then φ is equivalent to an $\exists \wedge FO^{k,+}$-sentence.

Corollary: We are done with this (bounded arity).